Lesson 34

1. How can orthogonal functions be generated?
   Given a linearly independent sequence of functions \( \{\phi_j\} \) it is always possible to construct an orthogonal sequence of functions from them - using the Gramm Schmidt procedure. Orthogonal functions are also generated by the eigen functions of a self adjoint system.

2. Does the “best approximation” always exist and is it unique?
   Given a set of orthogonal basis functions, the best approximation for that basis always exists and is unique. The coefficients \( c_j^* \) corresponding to the best approximation are given by:
   \[
   c_j^* = \frac{(\phi_j, f)}{(\phi_j, \phi_j)} = \frac{(\phi_j, f)}{\|\phi_j\|^2}
   \]
   which exists since \( \|\phi_j\|^2 \neq 0 \) if \( \phi_j \neq 0 \) (since the inner product is positive definite).

3. What are Bessel’s inequality and Parseval’s formula?
   If \( f^* = \sum_{j=0}^{n-1} c_j^* \phi_j \) is the best approximation to \( f \), then Bessel's inequality states that
   \[
   \sum_{j=0}^{\infty} (c_j^*)^2 \|\phi_j\|^2 \leq \|f\|^2
   \]
   for an infinite number of basis functions. If the basis functions of the infinite series are bounded then \( \sum_{j=0}^{\infty} (c_j^*)^2 \|\phi_j\|^2 = \|f\|^2 \). This known as Parseval's formula.

4. What is the general recursion formula for orthogonal polynomials?
   For \( n \geq 1 \) all families of orthogonal polynomials satisfy a three term recursion formula which allows a new member of the family, \( \phi_{n+1}(x) \) to be generated from existing members \( \phi_n(x) \) and \( \phi_{n-1}(x) \). The recursion formula enables \( \phi_{n+1}(x) \) to be determined uniquely, up to an arbitrary constant \( \alpha_n \) which relates the leading coefficient of \( \phi_{n+1}(x) \), \( \gamma_{n+1} \) to the leading coefficient of \( \phi_n(x) \), \( \gamma_n \).

   The recursion formula allows construction of a series of orthogonal polynomials in unique fashion if the first two terms of the series are known.
5. **How many zeros does an orthogonal polynomial of degree $n$ have? Are the zeros simple zeros?**

By construction the $n^{th}$ order polynomial has $n$ zeros. However, in addition, a $n^{th}$ degree polynomial in a family of orthogonal polynomials with weight function $w$ on an interval $[a,b]$ has $n$ simple zeros, all of which lie in $[a,b]$.

6. **What are Legendre polynomials?**

The Legendre polynomials are a series of orthogonal polynomials which are all roots of Legendre's equation. They are defined by the formula:

$$P_0(x) = 1 \quad P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}[(x^2 - 1)^n], \quad (n = 1, 2, \ldots)$$

The inner product, defined over $[-1,1]$, has weight factor 1.

Thus: $(P_n, P_j) = \int_{-1}^{1} \frac{d^n}{dx^n}(x^2 - 1)^n \frac{d^j}{dx^j}(x^2 - 1)^j \, dx = 0$ if $n \neq j$.  

$$= \frac{2}{2n + 1} \text{ if } n = j.$$  

The Legendre polynomials also satisfy symmetry: $P_n(x) = (-1)^n P_n(-x)$ similar to Chebyshev polynomials.