The triangular elements with different numbers of nodes are used for solving two dimensional solid members. The linear triangular element was the first type of element developed for the finite element analysis of 2D solids. However, it is observed that the linear triangular element is less accurate compared to linear quadrilateral elements. But the triangular element is still a very useful element for its adaptivity to complex geometry. These are used if the geometry of the 2D model is complex in nature. Constant strain triangle (CST) is the simplest element to develop mathematically. In CST, strain inside the element has no variation (Ref. module 3, lecture 2) and hence element size should be small enough to obtain accurate results. As indicated earlier, the displacement is expressed in two orthogonal directions in case of 2D solid elements. Thus the displacement field can be written as

\[
\{d\} = \begin{bmatrix} u \\ v \end{bmatrix}
\]

Here, \(u\) and \(v\) are the displacements parallel to \(x\) and \(y\) directions respectively.

### 5.1.1 Element Stiffness Matrix for CST

A typical triangular element assumed to represent a subdomain of a plane body under plane stress/strain condition is represented in Fig. 5.1.1. The displacement \((u, v)\) of any point \(P\) is represented in terms of nodal displacements

\[
\begin{align*}
  u &= N_1 u_1 + N_2 u_2 + N_3 u_3 \\
  v &= N_1 v_1 + N_2 v_2 + N_3 v_3
\end{align*}
\]

Where, \(N_1, N_2, N_3\) are the shape functions as described in module 3, lecture 2.
The strain-displacement relationship for two dimensional plane stress/strain problem can be simplified in the following form from three dimensional cases (eq.1.3.9 to 1.3.14).

\[
\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2} \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial x} \right)^2 \right]
\]

\[
\varepsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2} \left[ \left( \frac{\partial u}{\partial y} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 \right]
\]

\[
\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial y} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)
\]

(5.1.3)

In case of small amplitude of displacement, one can ignore the nonlinear term of the above equation and will reach the following expression.

\[
\varepsilon_x = \frac{\partial u}{\partial x}
\]

\[
\varepsilon_y = \frac{\partial v}{\partial y}
\]

\[
\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}
\]

(5.1.4)

Hence the element strain components can be represented as,

\[
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial u}{\partial x} & \frac{\partial N_1}{\partial x} u_1 + \frac{\partial N_2}{\partial x} u_2 + \frac{\partial N_3}{\partial x} u_3 \\
\frac{\partial v}{\partial y} & \frac{\partial N_1}{\partial y} v_1 + \frac{\partial N_2}{\partial y} v_2 + \frac{\partial N_3}{\partial y} v_3 \\
\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} & \frac{\partial u}{\partial y} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
u_3
\end{bmatrix}
\]

Or,

\[
\varepsilon = \begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} \\
0 & 0 & 0 \\
\frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} \\
\frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} \\
\frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x}
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
u_3
\end{bmatrix}
\]

(5.1.5)

Or,

\[
\varepsilon = [B]\{d\}
\]

(5.1.6)
In the above equation \([B]\) is called as strain displacement relationship matrix. The shape functions for the 3 node triangular element in Cartesian coordinate is represented as,

\[
\begin{bmatrix}
N_1 \\
N_2 \\
N_3
\end{bmatrix} = \frac{1}{2A} \begin{bmatrix}
(x_2 y_3 - x_3 y_2) + (y_2 - y_3) x + (x_3 - x_2) y \\
(x_3 y_1 - x_1 y_3) + (y_3 - y_1) x + (x_1 - x_3) y \\
(x_1 y_2 - x_2 y_1) + (y_1 - y_2) x + (x_2 - x_1) y
\end{bmatrix}
\]

Or,

\[
\begin{bmatrix}
N_1 \\
N_2 \\
N_3
\end{bmatrix} = \frac{1}{2A} \begin{bmatrix}
[\alpha_1 + \beta_1 x + \gamma_1 y] \\
[\alpha_2 + \beta_2 x + \gamma_2 y] \\
[\alpha_3 + \beta_3 x + \gamma_3 y]
\end{bmatrix}
\]

Where,

\[
\begin{align*}
\alpha_1 &= (x_2 y_3 - x_3 y_2), & \alpha_2 &= (x_3 y_1 - x_1 y_3), & \alpha_3 &= (x_1 y_2 - x_2 y_1), \\
\beta_1 &= (y_2 - y_3), & \beta_2 &= (y_3 - y_1), & \beta_3 &= (y_1 - y_2), \\
\gamma_1 &= (x_3 - x_2), & \gamma_2 &= (x_2 - x_1), & \gamma_3 &= (x_2 - x_1),
\end{align*}
\]

Hence the required partial derivatives of shape functions are,

\[
\begin{align*}
\frac{\partial N_1}{\partial x} &= \frac{\beta_1}{2A}, & \frac{\partial N_2}{\partial x} &= \frac{\beta_2}{2A}, & \frac{\partial N_3}{\partial x} &= \frac{\beta_3}{2A}, \\
\frac{\partial N_1}{\partial y} &= \frac{\gamma_1}{2A}, & \frac{\partial N_2}{\partial y} &= \frac{\gamma_2}{2A}, & \frac{\partial N_3}{\partial y} &= \frac{\gamma_3}{2A}.
\end{align*}
\]

Hence the value of \([B]\) becomes:

\[
[B] = \begin{bmatrix}
\frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} \\
\frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x}
\end{bmatrix}
\]

Or,

\[
[B] = \frac{1}{2A} \begin{bmatrix}
\beta_1 & \beta_2 & \beta_3 & 0 & 0 & 0 \\
0 & 0 & 0 & \gamma_1 & \gamma_2 & \gamma_3 \\
\gamma_1 & \gamma_2 & \gamma_3 & \beta_1 & \beta_2 & \beta_3
\end{bmatrix}
\]

(5.1.9)
According to Variational principle described in module 2, lecture 1, the stiffness matrix is represented as,

\[
[k] = \iiint_{V} [B]^T [D] [B] \, d\Omega 
\]  

(5.1.10)

Since, \([B]\) and \([D]\) are constant matrices; the above expression can be expressed as

\[
[k] = [B]^T [D] [B] \iiint_{V} dV = [B]^T [D] [B] V 
\]  

(5.1.11)

For a constant thickness \((t)\), the volume of the element will become \(A.t\). Hence the above equation becomes,

\[
[k] = [B]^T [D] [B] A t 
\]  

(5.1.12)

For plane stress condition, \([D]\) matrix will become:

\[
[D] = \frac{E}{1 - \mu^2} \begin{bmatrix}
1 & \mu & 0 \\
\mu & 1 & 0 \\
0 & 0 & \frac{1 - \mu}{2}
\end{bmatrix}
\]  

(5.1.13)

Therefore, for a plane stress problem, the element stiffness matrix becomes,

\[
[k] = \frac{Et}{4A(1 - \mu^2)} \begin{bmatrix}
\beta_1 & 0 & \gamma_1 \\
\beta_2 & 0 & \gamma_2 \\
\beta_3 & 0 & \gamma_3 \\
0 & \gamma_1 & \beta_1 \\
0 & \gamma_2 & \beta_2 \\
0 & \gamma_3 & \beta_3
\end{bmatrix}
\begin{bmatrix}
1 & \mu & 0 \\
\mu & 1 & 0 \\
0 & 0 & 1 - \mu \\
0 & 0 & 0 \\
\gamma_1 & \gamma_2 & \gamma_3 \\
\beta_1 & \beta_2 & \beta_3
\end{bmatrix}
\]  

(5.1.14)

Or,

\[
[k] = \frac{Et}{4A(1 - \mu^2)} \begin{bmatrix}
\beta_1^2 + C\gamma_1^2 & \beta_1\beta_2 + C\gamma_1\gamma_2 & \beta_1\beta_3 + C\gamma_1\gamma_3 & \frac{(1 + \mu)}{2}\beta_1\gamma_1 & \mu\beta_2\gamma_1 + C\beta_1\gamma_1 & \mu\beta_3\gamma_1 + C\beta_1\gamma_1 \\
\beta_2^2 + C\gamma_2^2 & \beta_2\beta_1 + C\gamma_2\gamma_1 & \beta_2\beta_3 + C\gamma_2\gamma_3 & \frac{(1 + \mu)}{2}\beta_2\gamma_2 & \mu\beta_3\gamma_2 + C\beta_2\gamma_2 & \mu\beta_1\gamma_2 + C\beta_2\gamma_2 \\
\beta_3^2 + C\gamma_3^2 & \beta_3\beta_1 + C\gamma_3\gamma_1 & \beta_3\beta_2 + C\gamma_3\gamma_2 & \frac{(1 + \mu)}{2}\beta_3\gamma_3 & \mu\beta_1\gamma_3 + C\beta_3\gamma_3 & \mu\beta_2\gamma_3 + C\beta_3\gamma_3 \\
\gamma_1^2 + C\beta_1^2 & \gamma_1\gamma_2 + C\beta_1\beta_2 & \gamma_1\gamma_3 + C\beta_1\beta_3 & \frac{(1 + \mu)}{2}\gamma_1\beta_1 & \mu\gamma_2\beta_1 + C\gamma_1\beta_1 & \mu\gamma_3\beta_1 + C\gamma_1\beta_1 \\
\gamma_2^2 + C\beta_2^2 & \gamma_2\gamma_1 + C\beta_2\beta_1 & \gamma_2\gamma_3 + C\beta_2\beta_3 & \frac{(1 + \mu)}{2}\gamma_2\beta_2 & \mu\gamma_3\beta_2 + C\gamma_2\beta_2 & \mu\gamma_1\beta_2 + C\gamma_2\beta_2 \\
\gamma_3^2 + C\beta_3^2 & \gamma_3\gamma_1 + C\beta_3\beta_1 & \gamma_3\gamma_2 + C\beta_3\beta_2 & \frac{(1 + \mu)}{2}\gamma_3\beta_3 & \mu\gamma_1\beta_3 + C\gamma_3\beta_3 & \mu\gamma_2\beta_3 + C\gamma_3\beta_3 \\
\end{bmatrix}
\]  

(5.1.15)

Where, \(C = \frac{(1 - \mu)}{2}\)

Similarly for plane strain condition, \([D]\) matrix is equal to,
\[ [D] = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} (1-\mu) & \mu & 0 \\ \mu & (1-\mu) & 0 \\ 0 & 0 & \frac{1-2\mu}{2} \end{bmatrix} \]  

(5.1.16)

Hence the element stiffness matrix will become:

\[
[k] = \frac{E_t}{2A(1+\mu)} \begin{bmatrix}
M\beta_1^2 + \gamma_1^2 & M\beta_1\beta_2 + \gamma_1\gamma_2 & M\beta_1\beta_3 + \gamma_1\gamma_3 \\
M\beta_2^2 + \gamma_2^2 & M\beta_2\beta_3 + \gamma_2\gamma_3 & (\mu+1)\beta_2\gamma_1 + \beta_2\gamma_3 \\
M\beta_3^2 + \gamma_3^2 & M\beta_3\beta_1 + \gamma_3\gamma_1 & (\mu+1)\beta_3\gamma_2 + \beta_3\gamma_1 \\
\end{bmatrix}
\]

(5.1.17)

Where \( M = (1-\mu) \)

### 5.1.2 Nodal Load Vector for CST

From the principle of virtual work,

\[
\int_\Omega \delta \{\varepsilon\}^T \{\sigma\} \mathrm{d}\Omega = \int_\Gamma \delta \{u\}^T \{F_\Gamma\} \mathrm{d}\Gamma + \int_\Omega \delta \{u\}^T \{F_\Omega\} \mathrm{d}\Omega
\]

(5.1.18)

Where, \( F_\Gamma \) and \( F_\Omega \) are the surface and body forces respectively. Using the relationship between stress-stain and strain displacement, one can derive the following expressions:

\[
\{\sigma\} = [D][B] \{d\}, \quad \delta \{\varepsilon\} = [B] \delta \{d\} \quad \text{and} \quad \delta \{u\} = [N] \delta \{d\}
\]

(5.1.19)

Hence eq. (5.1.18) can be rewritten as,

\[
\int_\Omega \delta \{d\}^T [B]^T [D][B] \{d\} \mathrm{d}\Omega = \int_\Gamma \delta \{d\}^T [N]^S_T \{F_\Gamma\} \mathrm{d}\Gamma + \int_\Omega \delta \{d\}^T [N]^B_T \{F_\Omega\} \mathrm{d}\Omega
\]

(5.1.20)

Or,

\[
\int_\Omega [B]^T [D][B] \{d\} \mathrm{d}\Omega = \int_\Gamma [N]^S_T \{F_\Gamma\} \mathrm{d}\Gamma + \int_\Omega [N]^B_T \{F_\Omega\} \mathrm{d}\Omega
\]

(5.1.21)

Here, \([N]^S\) is the shape function along the boundary where forces are prescribed. Eq.(5.1.21) is equivalent to \([k]\{d\} = \{F\}\), and thus, the nodal load vector becomes

\[
\{F\} = \int_\Gamma [N]^S_T \{F_\Gamma\} \mathrm{d}\Gamma + \int_\Omega [N]^B_T \{F_\Omega\} \mathrm{d}\Omega
\]

(5.1.22)

For a constant thickness of the triangular element eq.(5.1.22) can be rewritten as

\[
\{F\} = t \int_S [N]^S_T \{F_\Gamma\} \mathrm{d}s + t \int_A [N]^B_T \{F_\Omega\} \mathrm{d}A
\]

(5.1.23)

For the a three node triangular two dimensional element, one can represent \( F_\Omega \) and \( F_\Gamma \) as,
\{F_\Omega\} = \begin{bmatrix} F_{\Omega x} \\ F_{\Omega y} \end{bmatrix} \quad \text{and} \quad \{F_T\} = \begin{bmatrix} F_{T x} \\ F_{T y} \end{bmatrix}

For example, in case of gravity load on CST element, \(\{F_\Omega\} = \begin{bmatrix} F_{\Omega x} \\ F_{\Omega y} \end{bmatrix} = \begin{bmatrix} 0 \\ -\rho g \end{bmatrix}\)

For this case, the shape functions in terms of area coordinates are:

\[
[N] = \begin{bmatrix} L_1 & L_2 & L_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & L_1 & L_2 & L_3 \end{bmatrix}
\]

As a result, the force vector on the element considering only gravity load, will become,

\[
\{F\} = t \int_A \begin{bmatrix} L_1 \\ L_2 \\ 0 \\ 0 \\ 0 \\ L_3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\rho g \end{bmatrix} dA = t \int_A \begin{bmatrix} 0 \\ 0 \\ 0 \\ -L_1 \rho g \\ -L_2 \rho g \\ -L_3 \rho g \end{bmatrix} dA = -\rho g t \int_A \begin{bmatrix} 0 \\ 0 \\ 0 \\ L_1 \\ L_2 \\ L_3 \end{bmatrix} dA
\]

The integration in terms of area coordinate is given by,

\[
\int_A L_1^p L_2^q L_3^r dA = \frac{p!q!r!}{(p+q+r+2)!} 2A
\]

Thus, the nodal load vector will finally become

\[
\{F\} = -\rho g t \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\rho g At \frac{1!0!0!}{(1+0+0+2)!} \\ -\rho g At \frac{0!1!0!}{(0+1+0+2)!} \\ -\rho g At \frac{0!0!1!}{(0+0+1+2)!} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = -\rho g At \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 3 \frac{2A}{1} \\ 3 \frac{2A}{1} \end{bmatrix}
\]