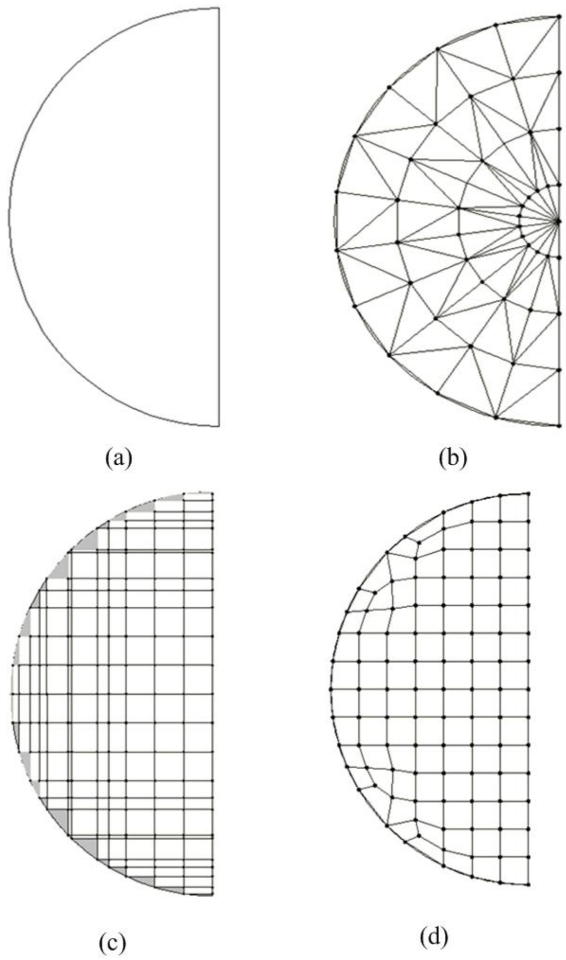
3.6.1 Necessity of Isoparametric Formulation

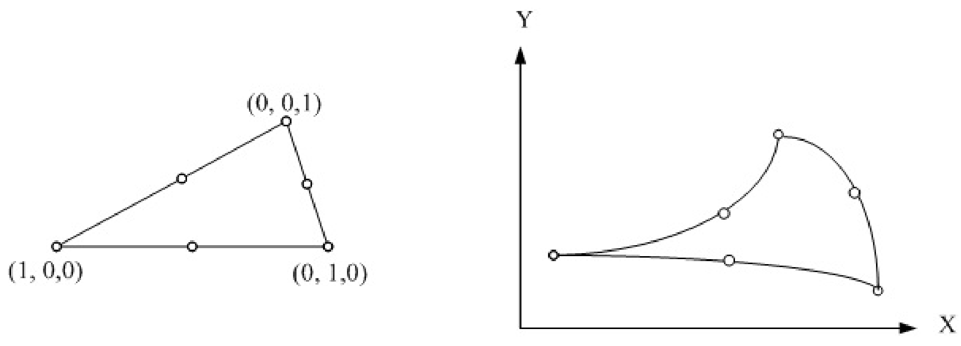
The two or three dimensional elements discussed till now are of regular geometry (e.g. triangular and rectangular element) having straight edge. Hence, for the analysis of any irregular geometry, it is difficult to use such elements directly. For example, the continuum having curve boundary as shown in the Fig. 3.6.1(a) has been discretized into a mesh of finite elements in three ways as shown.

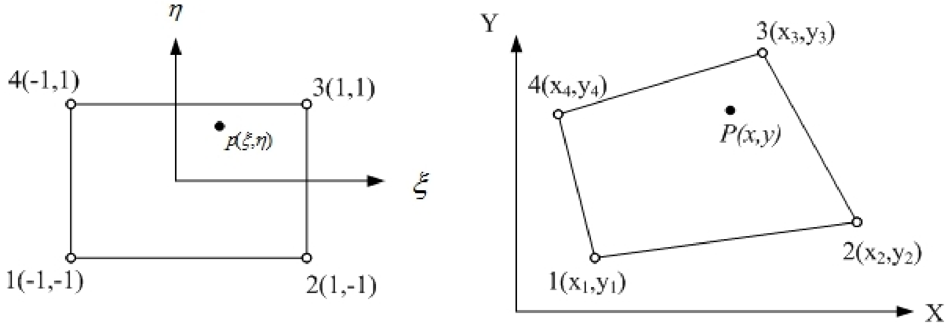


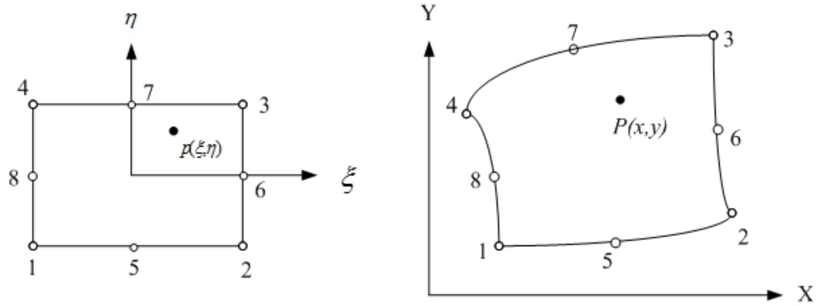
(a) The Continuum to be discritized (b) Discritization using Triangular Elements (c) Discritization using rectangular elements (d) Discritization using a combination of rectangular and quadrilateral elements

Fig 3.6.1 Discretization of a continuum using different elements

Figure 3.6.1(b) presents a possible mesh using triangular elements. Though, triangular elements can suitable approximate the circular boundary of the continuum, but the elements close to the center becomes slender and hence affect the accuracy of finite element solutions. One possible solution to the problem is to reduce the height of each row of elements as we approach to the center. But, unnecessary refining of the continuum generates relatively large number of elements and thus increases computation time. Alternatively, when meshing is done using rectangular elements as shown in Fig 3.6.1(c), the area of continuum excluded from the finite element model is significantlyadequate to provide incorrect results. In order to improve the accuracy of the result one can generate mesh using very small elements. But, this will significantly increase the computation time. Another possible way is to use a combination of both rectangular and triangular elements as discussed in section 3.2. But such types of combination may not provide the best solution in terms of accuracy, since different order polynomials are used to represent the field variables for different types of elements. Also the triangular elements may be slender and thus can affect the accuracy. In Fig.3.6.1(d), the same continuum is discritized with rectangular elements near center and with four-node quadrilateral elements near boundary. This four-node quadrilateral element can be derived from rectangular elements using the concept of mapping. Using the concept of mapping regular triangular, rectangular or solid elements in natural coordinate system (known as parent element) can be transformed into global Cartesian coordinate system having arbitrary shapes (with curved edge or surfaces). Fig.3.6.2 shows the parent elements in natural coordinate system and the mapped elements in global Cartesian system.




η
4(-1,1)
3(1,1)
p(ξ,η)
ξ
1(-1,-1)
2(1,-1)
Y
3(x3,y3)
4(x4,y4)
P(x,y)
2(x2,y2)
1(x1,y1)
X


η
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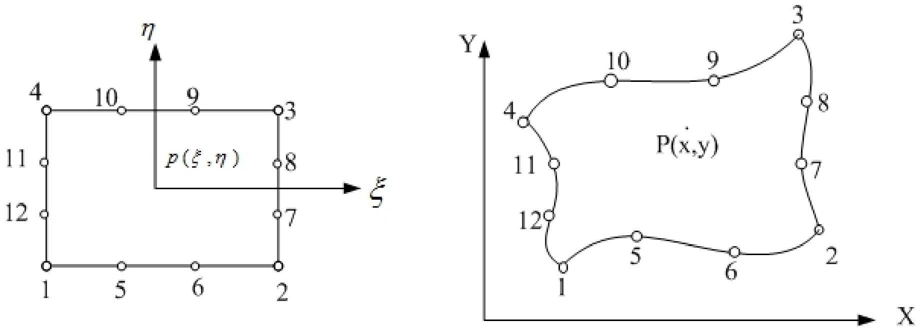
7
3
p(ξ,η)
8
6
ξ
1
5
2
Y
7
3
4
P(x,y)
6
8
2
1
5
X


η
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10
9
3
11
p(ξ,η)
8
12
7
ξ
1
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6
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Y
10
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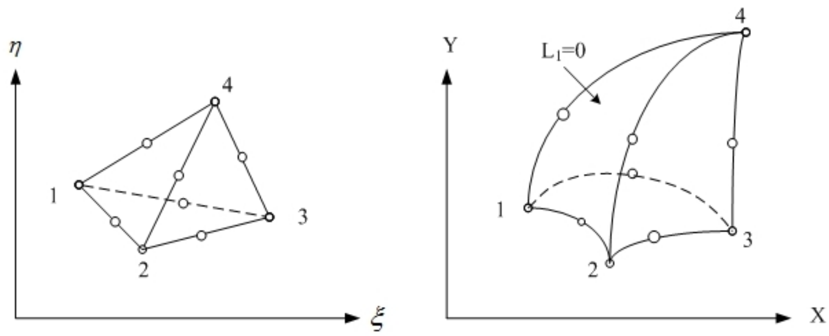
5
2
6
1
X


η
4
1
3
2
ξ
Y
4
L1=0
1
3
2
X

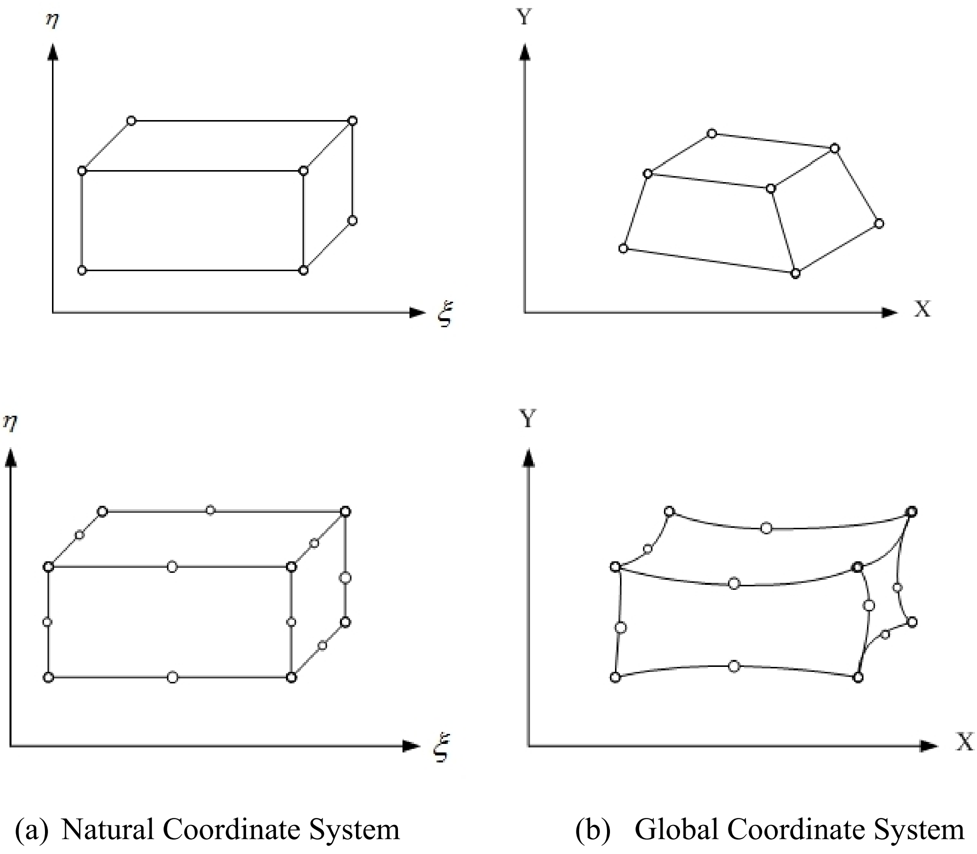


Fig. 3.6.2 Mapping of isoparametric elements in global coordinate system

3.6.2 Coordinate Transformation

The geometry of an element may be expressed in terms of the interpolation functions as follows.

$$x = N_1 x_1 + N_2 x_2 + \ldots + N_n x_n = \sum_{i=1}^{n} N_i x_i$$

$$y = N_1 y_1 + N_2 y_2 + \ldots + N_n y_n = \sum_{i=1}^{n} N_i y_i \qquad (3.6.1)$$

$$z = N_1 z_1 + N_2 z_2 + \ldots + N_n z_n = \sum_{i=1}^{n} N_i z_i$$

Where,

n=No. of Nodes

N_i=Interpolation Functions

x_i, y_i, z_i=Coordinates of Nodal Points of the Element

One can also express the field variable variation in the element as

$$\phi(\xi,\eta,\zeta)=\sum_{i=1}^{n}N_i(\xi,\eta,\zeta)\phi_i \quad (3.6.2)$$

As the same shape functions are used for both the field variableand description of element geometry, the method is known as isoparametricmapping. The element defined by such a method is known as an isoparametric element. This method can be used to transform the natural coordinates of a point to the Cartesian coordinate system and vice versa.

Example 3.6.1

Determine the Cartesian coordinate of the point P (ξ= 0.8, η= 0.9) as shown in Fig. 3.6.3.

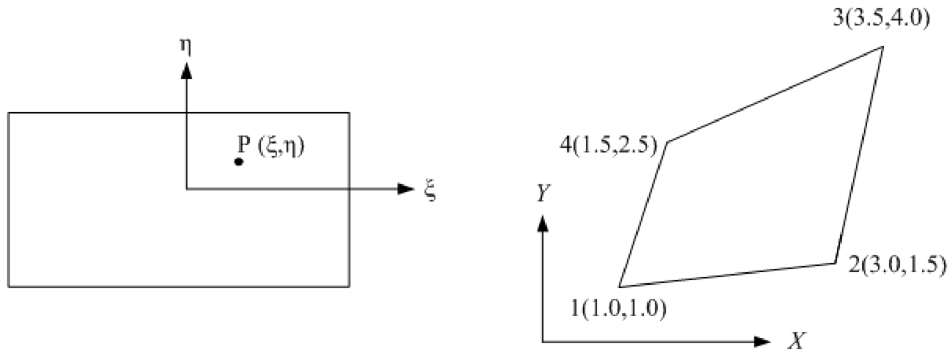


Fig. 3.6.3 Transformation of Coordinates

Solution:

As described above, the relation between two coordinate systems can be represented through their interpolation functions. Therefore, the valuesof the interpolation function at point P will be

$$N_1=\frac{(1-\xi)(1-n)}{4}=\frac{(1-0.8)(1-0.9)}{4}=0.005$$

$$N_2=\frac{(1+\xi)(1-n)}{4}=\frac{(1+0.8)(1-0.9)}{4}=0.045$$

$$N_3=\frac{(1+\xi)(1+n)}{4}=\frac{(1+0.8)(1+0.9)}{4}=0.855$$

$$N_4=\frac{(1-\xi)(1+n)}{4}=\frac{(1-0.8)(1+0.9)}{4}=0.095$$

Thus the coordinate of point *P* in Cartesian coordinate system can be calculated as

$$x = \sum_{i=1}^{4} N_i x_i = 0.005\times1 + 0.045\times3 + 0.855\times3.5 + 0.095\times1.5 = 3.275$$

$$y = \sum_{i=1}^{4} N_i y_i = 0.005\times1 + 0.045\times1.5 + 0.855\times4.0 + 0.095\times2.5 = 3.73$$

Thus the coordinate of point P (ξ= 0.8, η= 0.9) in Cartesian coordinate system will be 3.275, 3.73.

Solid isoparametric elements can easily be formulated by the extension of the procedure followed for 2-D elements. Regardless of the number of nodes or possible curvature of edges, the solid element is just like a plane element which is mapped into the space of natural co-ordinates, i.e, $\xi = \pm1, \eta = \pm1, \zeta = \pm1$.

3.6.3 Concept of Jacobian Matrix

A variety of derivatives of the interpolationfunctions with respect to the global coordinates are necessary to formulate the element stiffness matrices. As the both element geometry andvariation of the shape functions are represented in terms of the naturalcoordinates of the parent element,some additional mathematical obstacle arises. For example, in case of evaluation of the strain vector, the operator matrix is with respect to x and y, but the interpolation function is with ξ *and* η. Therefore, the operator matrix is to be transformed for taking derivative with ξ *and* η. The relationship between two coordinate systems may be computed by using the chain rule of partial differentiation as

$$\frac{\partial}{\partial \xi} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \xi} \quad and \quad \frac{\partial}{\partial \eta} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \eta} \tag{3.6.3}$$

The above equations can be expressed in matrix form as well.

$$\begin{Bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{Bmatrix} = [J] \begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{Bmatrix} \tag{3.6.4}$$

The matrix [*J*] is denoted as Jacobian matrix which is: $\begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}$.As we know, $x = \sum_{i=1}^{n} N_i x_i$

where, *n*is the number of nodes in an element. Hence, $J_{11} = \dfrac{\partial x}{\partial \xi} = \dfrac{\partial \sum_{i=1}^{n} N_i x_i}{\partial \xi} = \sum_{i=1}^{n} \dfrac{\partial N_i}{\partial \xi} x_i$

Similarly one can calculate the other terms J_{12}, $J_{21}$and J_{22} of the Jacobian matrix. Hence,

$$[J] = \begin{bmatrix} \sum_{i=1}^{n} \frac{\partial N_i}{\partial \xi} x_i & \sum_{i=1}^{n} \frac{\partial N_i}{\partial \xi} y_i \\ \sum_{i=1}^{n} \frac{\partial N_i}{\partial \eta} x_i & \sum_{i=1}^{n} \frac{\partial N_i}{\partial \eta} y_i \end{bmatrix} \tag{3.6.5}$$

From eq. (3.6.4), one can write

$$\begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix} \tag{3.6.6}$$

Considering $\begin{bmatrix} J_{11}^* & J_{12}^* \\ J_{21}^* & J_{22}^* \end{bmatrix}$ are the elements of inverted [J] matrix, we may arise into the following relations.

$$\begin{aligned} \frac{\partial}{\partial x} &= J_{11}^* \cdot \frac{\partial}{\partial \xi} + J_{12}^* \cdot \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial y} &= J_{21}^* \cdot \frac{\partial}{\partial \xi} + J_{22}^* \cdot \frac{\partial}{\partial \eta} \end{aligned} \tag{3.6.7}$$

Similarly, for three dimensional case, the following relation exists between the derivative operators in the global and the natural coordinate system.

$$\begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial \zeta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{Bmatrix} = [J] \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{Bmatrix} \tag{3.6.8}$$

Where,

$$[J]=\begin{bmatrix}\dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} & \dfrac{\partial z}{\partial \xi}\\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} & \dfrac{\partial z}{\partial \eta}\\ \dfrac{\partial x}{\partial \zeta} & \dfrac{\partial y}{\partial \zeta} & \dfrac{\partial z}{\partial \zeta}\end{bmatrix} \qquad (3.6.9)$$

[*J*] is known as the Jacobian Matrix for three dimensional case. Putting eq. (3.6.1) in eq. (3.6.9) and after simplifying one can get

$$[J]=\sum_{i=1}^{n}\begin{bmatrix}\dfrac{\partial N_i}{\partial \xi}x_i & \dfrac{\partial N_i}{\partial \xi}y_i & \dfrac{\partial N_i}{\partial \xi}z_i\\ \dfrac{\partial N_i}{\partial \eta}x_i & \dfrac{\partial N_i}{\partial \eta}y_i & \dfrac{\partial N_i}{\partial \eta}z_i\\ \dfrac{\partial N_i}{\partial \zeta}x_i & \dfrac{\partial N_i}{\partial \zeta}y_i & \dfrac{\partial N_i}{\partial \zeta}z_i\end{bmatrix} \qquad (3.6.10)$$

From eq. (3.6.8), one can find the following expression.

$$\begin{Bmatrix}\dfrac{\partial}{\partial x}\\ \dfrac{\partial}{\partial y}\\ \dfrac{\partial}{\partial z}\end{Bmatrix}=[J]^{-1}\begin{Bmatrix}\dfrac{\partial}{\partial \xi}\\ \dfrac{\partial}{\partial \eta}\\ \dfrac{\partial}{\partial \zeta}\end{Bmatrix} \qquad (3.6.11)$$

Considering $[J]^{-1}=\begin{bmatrix}J_{11}^* & J_{12}^* & J_{13}^*\\ J_{21}^* & J_{22}^* & J_{23}^*\\ J_{31}^* & J_{32}^* & J_{33}^*\end{bmatrix}$ we can arrived at the following relations.

$$\begin{aligned}\frac{\partial}{\partial x}&=J_{11}^*\cdot\frac{\partial}{\partial \xi}+J_{12}^*\cdot\frac{\partial}{\partial \eta}+J_{13}^*\cdot\frac{\partial}{\partial \zeta}\\ \frac{\partial}{\partial y}&=J_{21}^*\cdot\frac{\partial}{\partial \xi}+J_{22}^*\cdot\frac{\partial}{\partial \eta}+J_{23}^*\cdot\frac{\partial}{\partial \zeta}\\ \frac{\partial}{\partial z}&=J_{31}^*\cdot\frac{\partial}{\partial \xi}+J_{32}^*\cdot\frac{\partial}{\partial \eta}+J_{33}^*\cdot\frac{\partial}{\partial \zeta}\end{aligned} \qquad (3.6.12)$$