Module 2
WAVE PROPAGATION
(Lectures 7 to 9)

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Lecture 7

Wave Propagation

Topics

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2.1 INTRODUCTION

It is the continuous nature of geologic materials that cause soil dynamics and geotechnical earthquake engineering to diverge from their structural counterparts. While most structures can readily be idealized as assemblages of discrete masses with discrete sources of stiffness, geologic materials cannot. They must be treated as continua, and their response to dynamic disturbance must be described in the context of wave propagation.

Some basic concepts of wave propagation have been alluded to in previous module; a more fundamental treatment of the basic concepts is presented in this module. The presentation follows a repeated pattern of simple-to-complex applications. The relatively simple problem of waves in unbounded media is followed by the more complicated problem of waves in bounded and layered media. Within each, the concepts are presented first for the simple case of one-dimensional wave propagation, and then for the more general three-dimensional case. The careful reader will note that the basic techniques and principles used to solve the more complicated cases the generally the same as those used for the simple cases; the additional complexity simply results from the need to consider more dimensions.
2.2 WAVE IN UNBOUND MEDIA

The propagation of stress waves is most easily understood by first considering an unbounded, or “infinite”, medium [i.e., one that extends infinitely in the direction(s) of wave propagation]. A simple, one-dimensional idealization of an unbounded medium is that of an infinitely long rod or bar. Using the basic requirements of equilibrium of forces and compatibility of displacements, and using strain-displacement and stress-strain relationships, one-dimensional wave equation can be derived and solved. The process can be repeated, using the same requirements and relationships, for the more general case of wave propagation in a medium that extends infinitely in three orthogonal directions.

2.2.1 One-Dimensional wave propagation

Three different types of vibration can occur in a thin rod: longitudinal vibration during which the axis of the rod extends and contracts without lateral displacement; torsional vibration in which the rod rotates about it axis without lateral displacement of the axis; and flexural vibration during which the axis itself moves laterally. The flexural vibration problem has little application in soil dynamics and will not be considered further. For the first two cases, however, the operative wave equations are easily derived and solved.

2.2.2 Longitudinal waves in an infinitely long rod

Consider the free vibrations of an infinitely long, linear elastic, constrained rod with cross-sectional area $A$, Young’s modulus $E$, Poisson’s ratio $v$, and density $p$, as shown in (figure 2.1). If the rod is constrained against radial straining, then particle displacements caused by a longitudinal wave must be parallel to the axis of the rod. Assume that cross-sectional planes will remain planar and that stresses will be distributed uniformly over each cross section. As a stress wave travels along the rod and passes through the small element shown in (figure 2.2), the axial stress at the left end of the element ($x = x_0$) is $\sigma x_0$. At the right end ($x = x_0 + dx$), the axial stress is $\sigma x_0 + (\partial \sigma_x / \partial x) dx$. Then dynamic equilibrium of the element requires that where $u$ is the displacement in the $x$-direction.

![Figure 2.1](image)

**Figure 2.1** Constrained infinite rod for one-dimensional wave propagation. Constraint against radial straining schematically represented by rollers
This simply states that the unbalanced external forces acting on the ends of the element [the left side of (equation 2.1)] must equal the inertial force induced by acceleration of the mass of the element (the right side). Simplifying yields the one-dimensional equation of motion

\[ \frac{\partial \sigma_x}{\partial x} + \frac{\partial^2 u}{\partial x^2} dx \cdot A - \sigma x_0 A = \rho A dx \frac{\partial^2 u}{\partial t^2} \]  

(2.1)

In this form, the equation of motion is valid for any stress-strain behavior but cannot be solved directly because it mixes stresses (on the left side of equation 2.2) with displacements (on the right side). To simplify the equation of motion, the left side can be expressed in terms of displacement by using the stress-strain relationship, \( \sigma_x = M \varepsilon_x \), where the constrained modulus \( M = (1 - V)/(1 + V)(1 - 2V)E \), and the strain-displacement relationship, \( \varepsilon_x = \partial u/\partial x \). These substitutions allow the one-dimensional equation of motion to be written in the familiar form of the one-dimensional longitudinal wave equation for a constrained rod:

\[ \frac{\partial^2 u}{\partial t^2} = \frac{M}{\rho} \frac{\partial^2 u}{\partial x^2} \]  

(2.3)

The one-dimensional wave equation can be written in the alternative form where \( v_p \) is the wave propagation velocity; for this case, the wave travels at \( v_p = \sqrt{M/\rho} \).

\[ \frac{\partial^2 u}{\partial t^2} = v_p^2 \frac{\partial^2 u}{\partial x^2} \]  

(2.4)

Note that the wave propagation velocity depends only on the properties of the rod material (its stiffness and density) and is independent of the amplitude of the stress wave. The wave propagation velocity increases with increasing stiffness and with decreasing density. The wave propagation velocity is an extremely important material property that is relied upon heavily in soil dynamics and geotechnical earthquake engineering.

The wave propagation velocity is the velocity at which a stress wave would travel along the rod. It is not the same as the particle velocity, which is the velocity at which a single point within the rod would move as the wave passes through it. Knowing that \( \partial u = \varepsilon_x \partial x \) (from the strain-displacement relationship), \( \varepsilon_x = \sigma_x / M \).
(from the stress-strain relationship), and \( \partial x = v_p \partial t \) (from the definition of wave propagation velocity), the particle velocity \( \dot{u} \) can be shown to be

\[
\dot{u} = \frac{\partial u}{\partial t} = \varepsilon_x \frac{\partial x}{\partial t} = \frac{\sigma_x}{M} \frac{\partial t}{\partial t} = \frac{\sigma_x}{\rho v_p^2} v_p = \frac{\sigma_e}{\rho v_p^2}
\]

(Equation 2.5) shows that the particle velocity is proportional to the axial stress in the rod. The coefficient of proportionality, \( \rho v_p \), called the specific impedance of the material. The specific impedance is another important property that influences the behavior of waves at boundaries.

**Example 1**

Compute \( v_p \) for steel, vulcanized rubber, and water.

**Solution**

The constrained moduli and specific gravities of steel, rubber, and water can be found in a number of reference books; typical values are summarized below:

<table>
<thead>
<tr>
<th>MATERIAL</th>
<th>SPECIFIC GRAVITY, SG</th>
<th>M (PSI)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Steel</td>
<td>7.85</td>
<td>40.4 \times 10^6</td>
</tr>
<tr>
<td>Vulcanized Rubber</td>
<td>1.2</td>
<td>1.67 \times 10^6</td>
</tr>
<tr>
<td>Water</td>
<td>1.0</td>
<td>0.34 \times 10^6</td>
</tr>
</tbody>
</table>

With this information, the \( v_p \) values can be calculated from:

\[
v_p = \frac{M}{\sqrt{\rho}} = \frac{M}{\sqrt{(SG) \rho_w}} = \frac{M}{\sqrt{(SG) \gamma_w}}
\]

For steel,

\[
v_p = \sqrt{\frac{(40.4 \times 10^6 \text{ psi})(144 \text{ in}^2/\text{ft}^2)(32.2 \text{ ft/sec}^2)}{(7.85)(62.4 \text{ pcf})}} = 19556 \text{ ft/sec}
\]

For vulcanized rubber,

\[
v_p = \sqrt{\frac{(167 \times 10^6 \text{ psi})(144 \text{ in}^2/\text{ft}^2)(32.2 \text{ ft/sec}^2)}{(1.20)(62.4 \text{ pcf})}} = 101691 \text{ ft/sec}
\]

For water,

\[
v_p = \sqrt{\frac{(0.34 \times 10^6 \text{ psi})(144 \text{ in}^2/\text{ft}^2)(32.2 \text{ ft/sec}^2)}{(1.00)(62.4 \text{ pcf})}} = 5026 \text{ ft/sec}
\]

**2.2.3 Torsional waves in an infinity long rod**

Torsional waves involve rotation of the rod about its own axis. In the case of the longitudinal wave, the direction of particle motion was parallel to the direction of wave propagation. For torsional waves, particle motion is constrained to planes perpendicular to the direction of wave propagation. Development of a wave equation...
for torsional vibration, however, follows exactly the same steps as for longitudinal vibration. Consider the short segment of a cylindrical rod shown in (figure 2.3) as a torsional wave of torque amplitude $T$ travels along the road. Dynamic torsional equilibrium requires that the unbalanced eternal torque (left side of equation 2.6) is equal to the inertial torque (right side): where $J$ is the polar moment of inertia of the rod about its axis.

![Figure 2.3 Torque and rotation at end of element of length $dx$ and cross-sectional area, $A$](image)

\[
\left( T_{x_0} + \frac{\partial T}{\partial x} dx \right) - T_{x_0} = \rho J \frac{\partial^2 \theta}{\partial t^2}
\]  

(2.6)

This equilibrium equation can be simplified to produce the equation of motion

\[
\frac{\partial T}{\partial x} \rho J \frac{\partial^2 \theta}{\partial t^2}
\]  

(2.7)

Now, incorporating the torque-relationship where $G$ is the *shear modulus* of the rod, the *torsional wave equation* can be written as

\[
T = GJ \frac{\partial \theta}{\partial x}
\]  

(2.8)

\[
\frac{\partial^2 \theta}{\partial t^2} = \frac{G}{J} \frac{\partial^2 \theta}{\partial x^2} = v_s^2 \frac{\partial^2 \theta}{\partial x^2}
\]  

(2.9)

Where $v_s = \sqrt{G/\rho}$ is the velocity of propagation of the torsional wave. Note that the form of the wave equation for torsional waves (equation 2.9) is identical to that for longitudinal waves (equation 2.3), but the wave propagation velocities are different. The wave propagation velocity depends both on the stiffness of the rod in the mode of deformation induced by the wave and on the material density but is independent of the amplitude of the stress wave.

**Example 2**

Compute $v_s$ for steel, vulcanized rubber, and water.
Solution
The shear moduli of steel, rubber, and water can be found in a number of reference books; typical values are summarized below:

<table>
<thead>
<tr>
<th>MATERIAL</th>
<th>G (PSI)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Steel</td>
<td>$11.5 \times 10^6$</td>
</tr>
<tr>
<td>Vulcanized Rubber</td>
<td>$0.167 \times 10^6$</td>
</tr>
<tr>
<td>Water</td>
<td>0</td>
</tr>
</tbody>
</table>

With the information, the $v_s$ values can be calculated from

$$v_s = \sqrt{\frac{G}{\rho}} = \sqrt{\frac{G g}{(5G) \gamma w}}$$

For steel,

$$v_s = \sqrt{\frac{(11.5 \times 10^6 \text{ psi})(144 \text{ in}^2/\text{ft}^2)(32.2 \text{ ft/sec}^2)}{(7.85)(62.4 \text{ pcf})}} = 10434 \text{ ft/sec}$$

For vulcanized rubber,

$$v_s = \sqrt{\frac{(0.167 \times 10^6 \text{ psi})(144 \text{ in}^2/\text{ft}^2)(32.2 \text{ ft/sec}^2)}{(1.20)(62.4 \text{ pcf})}} = 3216 \text{ ft/sec}$$

For water,

$$v_s = 0$$

The last result is obvious—as an inviscid fluid, water can produce no resistance to shear stresses and consequently cannot transmit torsional waves.

### 2.2.4 Solution of the One-Dimensional Equation of Motion

The one-dimensional wave equation is partial differential equation of the form where $v$ represents the wave propagation velocity corresponding to the type of stress wave of interest. The solution of such an equation can be written in the form where $f$ and $g$ can be any arbitrary functions of $(vt - x)$ and $(vt + x)$ that satisfy (equation 2.10).

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2} \quad (2.10)$$

$$u(x,t) = f(vt - x) + g(vt + x) \quad (2.11)$$

Note that the argument of $f$ remains constant when $x$ increases with time (at velocity $v$), and the argument of $g$ remains constant when $x$ decreases with time. Therefore, the solution of (equation 2.11) describes a displacement wave $[f(vt - x)]$ travelling at velocity $v$. I the positive $x$-direction and another $g(vt + x)]$ travelling at the same speed in the negative $x$-direction. It also implies that the shapes of the waves do not change with position or time.

If the rod is subjected to some steady-state harmonic stress $\sigma(t) = \sigma_0 \cos \omega$ where $\sigma_0$ is the stress wave amplitude and $\omega$ is the circular
frequency of the applied loading, the solution can be expressed using the wave number, \( k = \bar{\omega} / v \), in the form

\[
\begin{align*}
\quad u(x,t) &= A \cos(\bar{\omega}t - kx)B \cos(\bar{\omega}t + kx) \\
\end{align*}
\]  

(2.12)

here the first and second terms describe harmonic waves propagating in the positive and negative \( x \)-direction respectively. The wave number is related to the wavelength, \( \lambda \), of the motion by where \( T \) is the period of the applied loading (note that wave number is to wavelength as circular frequency is to period) and \( f = 1/T \).

\[
\lambda = vT = \frac{v}{f} = \frac{2\pi}{\bar{\omega}} \quad v = \frac{2\pi}{k} 
\]

(2.13)

Note that at a given frequency, the wave-length increase with increasing wave propagation velocity. (Equation 2.12) indicates that the displacement varies harmonically with respect to both time and position as illustrated in (Figure 2.4). (Equation 2.13) and Figure 2.4 show that the wave number is to the wavelength as the circular frequency is to the period of vibration. For a wave propagating in the positive \( x \)-direction only \((B = 0)\), differentiating \( u(x,t) \) twice with respect to \( x \) and twice with respect to \( t \) and substituting into the wave equation (equation 2.10) gives which reduces to the identity \( \bar{\omega} = kv \), thereby verifying (equation 2.12) as a solution to the wave equation.

![Particle displacements (a) as function of time, and (b) as function of position along the rod](image)

Figure 2.4 Particle displacements (a) as function of time, and (b) as function of position along the rod

\[
-\bar{\omega}^2A \cos(\bar{\omega}t - kx) = -v^2 k^2 A \cos(\bar{\omega}t - kx) 
\]

(2.14)

Using complex notation (appendix A), the equivalent form of the solution to the written as

\[
\begin{align*}
\quad u(x,t) &= Ce^{i(\bar{\omega}t-kx)} + De^{i(\bar{\omega}t+kx)} \\
\end{align*}
\]

(2.15)

This form of the solution can be verified in the same way as the trigonometric form.

Example 3
Calculate the wavelength of harmonic longitudinal and torsional waves travelling along constrained steel and vulcanized rubber rods. Assume that the waves are
harmonic at a frequency of 10Hz.

Solution
Using (equation 2.13) and the wave propagation velocities computed in Examples 1 and 2.

Longitudinal waves
Steel: \( \lambda = \frac{v_p}{f} = \frac{19556 \text{ ft/sec}}{10 \text{ sec}^{-1}} = 1956 \text{ ft} \)

Vulcanized rubber: \( \lambda = \frac{v_p}{f} = \frac{101691 \text{ ft/sec}}{10 \text{ sec}^{-1}} = 10169 \text{ ft} \)

Torsional waves
Steel: \( \lambda = \frac{v_s}{f} = \frac{10434 \text{ ft/sec}}{10 \text{ sec}^{-1}} = 1043 \text{ ft} \)

Vulcanized rubber: \( \lambda = \frac{v_s}{f} = \frac{3216 \text{ ft/sec}}{10 \text{ sec}^{-1}} = 322 \text{ ft} \)

2.2.5 Three-Dimensional Wave Propagation
The preceding discussion of wave propagation in rods illustrates some of the basic principles of wave propagation, but an infinite rod is hardly an adequate model for describing the propagation of seismic waves through the earth. Since the earth is three-dimensional and sources of seismic energy are three-dimensional, seismic must be described in terms of three-dimensional wave propagation.

Derivations of three-dimensional equations of motion follow the same steps as those used for one-dimensional propagation; the equations of motion are formulated from equilibrium consideration, stress-strain relationships, and strain-displacement relationships. In the three-dimensional case, however, the various relationships are more complex and the derivation more cumbersome. Brief reviews of three-dimensional stress and strain notation and three-dimensional stress-strain behavior will precede derivation of the equation of motion.

2.2.6 Review of Stress Notation
The stress at a point on some plane passing through a solid does not usually act normal to the plane but has both normal and shear components. Considering a small element with one occur at the center of an \( x - y - z \) Cartesian coordinate system (figure 2.5) a total nine components of stress will act on its faces. These stresses are denoted by \( \sigma_x, \sigma_y, \sigma_z, \) and so on, where the first and second letters in the subscript describe the direction of the stress itself and the axis perpendicular to the plane in which it acts. Thus \( \sigma_{xx}, \sigma_{yy}, \text{ and } \sigma_{zz} \) are normal stresses, while the other six components represent shear stresses. Moment equilibrium of the element requires that which means that only six independent components of stress are required to define the state of stress of the element completely. In some references, the notation \( \sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \text{ and } \tau_{xz} \) is used to describe \( \sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{yz}, \text{ and } \sigma_{xz}, \) respectively.
Figure 2.5 Stress notation for element of dimensions $dx$ by $dy$ by $dz$

\[ \sigma_{xy} = \sigma_{yx}, \quad \sigma_{xz} = \sigma_{zx}, \quad \sigma_{yz} = \sigma_{zy} \]

2.2.7 Review of Strain Notation

Component of strain are easily visualized by considering the two-dimensional strain in the $x - y$ plane shown in (figure 2.6). The point $P$, at coordinates $(x_0, y_0)$ is at one corner of the infinitesimal element $PQRS$ which has a square shape before deformation. After deformation, the infinitesimal element has been displaced, distorted, and rotated into the shape $P'Q'R'S$. From (figure 2.6) $\tan \alpha_1 = \frac{dv}{dx}$ and $\tan \alpha_2 = \frac{du}{dy}$, where $u$ and $v$ represent displacement in the $x$- and $y$-direction, respectively. The shear strain in the $x - y$ plane is given by $\varepsilon_{xy} = \alpha_1 + \alpha_2$. For small deformation, the angles may be taken equal to their tangents so that the relationship between the shear strain and the displacements is $\varepsilon_{xy} = \frac{dv}{dx} + \frac{du}{dy}$. The rotation of the element about the $z$-axis is given by $\Omega_z = (\alpha_1 + \alpha_2)/2$. Analogous definitions can be developed for the $x - y$ and $y - z$ planes. For the three-dimensional case, the strain-displacement relationships are defined by

\begin{align*}
\varepsilon_{xx} &= \frac{du}{dx} \quad \varepsilon_{yy} = \frac{dy}{dy} \quad \varepsilon_{zz} = \frac{dw}{dz} \\
\varepsilon_{xy} &= \frac{dv}{dx} + \frac{du}{dy} \quad \varepsilon_{yx} = \frac{dw}{dy} + \frac{dv}{dz} \quad \varepsilon_{zx} = \frac{du}{dz} + \frac{dw}{dx}
\end{align*}

(2.16) (2.17)

Rigid-body rotation about the $x$-, $y$-, and $z$-axes are given by the rotation-displacement relationships

\begin{align*}
\Omega_x &= \frac{1}{2} \left( \frac{dw}{dy} - \frac{dv}{dz} \right) \quad \Omega_y = \frac{1}{2} \left( \frac{du}{dz} - \frac{dw}{dx} \right) \quad \Omega_z = \frac{1}{2} \left( \frac{dv}{dx} - \frac{du}{dy} \right)
\end{align*}

(2.18)
The first three quantities $\varepsilon_{xx}, \varepsilon_{yy},$ and $\varepsilon_{zz}$, represent the extensional and compressional strain parallel to the $x$-, $y$-, and $z$-axes are called normal strains. The second three quantities, $\varepsilon_{xy}, \varepsilon_{yz},$ and $\varepsilon_{zx}$, represent the components of shear strain in the planes corresponding to their suffixes. These six quantities are the components of strain that correspond to the deformation at $P$. In some references, the notation $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz},$ and $\gamma_{xz}$ is used to describe $\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \varepsilon_{xy}, \varepsilon_{yz},$ and $\varepsilon_{xz}$, respectively.

2.2.8 Review of Stress-Strain Relationships

Stresses and strains proportional in a linear elastic body. The stress-strain relationship can be described by Hooke’s laws, which can be written in generalized form as where the 36 coefficients represent the elastic constants of the material.

$$
s_{xx} = C_{11}\varepsilon_{xx} + C_{12}\varepsilon_{yy} + C_{13}\varepsilon_{zz} + C_{14}\varepsilon_{xy} + C_{15}\varepsilon_{yz} + C_{16}\varepsilon_{zx}
$$

$$
s_{yy} = C_{21}\varepsilon_{xx} + C_{22}\varepsilon_{yy} + C_{23}\varepsilon_{zz} + C_{24}\varepsilon_{xy} + C_{25}\varepsilon_{yz} + C_{26}\varepsilon_{zx}
$$

$$
s_{zz} = C_{31}\varepsilon_{xx} + C_{32}\varepsilon_{yy} + C_{33}\varepsilon_{zz} + C_{34}\varepsilon_{xy} + C_{35}\varepsilon_{yz} + C_{36}\varepsilon_{zx}
$$

(2.19)

$$
s_{xy} = C_{41}\varepsilon_{xx} + C_{42}\varepsilon_{yy} + C_{43}\varepsilon_{zz} + C_{44}\varepsilon_{xy} + C_{45}\varepsilon_{yz} + C_{46}\varepsilon_{zx}
$$

$$
s_{yz} = C_{51}\varepsilon_{xx} + C_{52}\varepsilon_{yy} + C_{53}\varepsilon_{zz} + C_{54}\varepsilon_{xy} + C_{55}\varepsilon_{yz} + C_{56}\varepsilon_{zx}
$$

$$
s_{zx} = C_{61}\varepsilon_{xx} + C_{62}\varepsilon_{yy} + C_{63}\varepsilon_{zz} + C_{64}\varepsilon_{xy} + C_{65}\varepsilon_{yz} + C_{66}\varepsilon_{zx}
$$

The requirement that the elastic strain energy must be unique function of the strain (which requires that $c_{ij} = c_{ji}$ for all $i$ and $j$) reduces the number of independent coefficients to 21. If the material is isotropic, the coefficients must be independent of direction, so that and all other constants are zero.

$$
C_{12} = C_{21} = C_{13} = C_{31} = C_{23} = C_{32} = \lambda
$$
Hooke’s law for an isotropic, linear, elastic material allows all components of stress and strain to be expressed in terms of the two Lame constants, \( \lambda \) and \( \mu \):

Where the volumetric strain \( \varepsilon = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} \).

\[
\begin{align*}
\sigma_{xx} &= \lambda \varepsilon + 2\mu \varepsilon_{xx} & \sigma_{xy} &= \mu \varepsilon_{xy} \\
\sigma_{yy} &= \lambda \varepsilon + 2\mu \varepsilon_{yy} & \sigma_{yz} &= \mu \varepsilon_{yz} \\
\sigma_{zz} &= \lambda \varepsilon + 2\mu \varepsilon_{zz} & \sigma_{zx} &= \mu \varepsilon_{zx}
\end{align*}
\] (2.21)

Note that the symbol \( \lambda \) is the used universally for both Lame’s constant and for wavelength; the context in which it is used should make its meaning obvious.

For convenience, several other parameters are often used to describe the stress-strain behavior of isotropic, linear, elastic materials, each of which can be expressed in terms of Lame’s constants. Some of the more common of these are:

- Young’s modulus: \( E = \frac{\mu (3\lambda + 2\mu)}{\lambda + \mu} \) (2.22.a)
- Bulk modulus: \( K = \lambda + \frac{2\mu}{3} \) (2.22.b)
- Shear modulus: \( G = \mu \) (2.22.c)
- Poisson’s ratio: \( \nu = \frac{\lambda}{2(\lambda + \mu)} \) (2.22.d)

Hooke’s law for an isotropic, linear, elastic material can be expressed using any combination of two of these parameters and/or Lambe’s constants.

### 2.2.9 Equation of Motion for a Three-Dimensional Elastic Solid

The three-dimensional equations of motion for an elastic solid are obtained from equilibrium requirements in much the same way as for the one-dimensional rod, except that equilibrium must be ensured in three perpendicular directions. Consider the variation in stress across an infinitesimal cube aligned with its sides parallel to the \( x - y - z \) axes shown in (Figure 2.7). Assuming that the average stress on each face of the cube is represented by the stress shown at the center of the face, the resultant forces acting in the \( x-, y- \) and \( z- \) directions can be evaluated. In the \( x- \) direction the unbalanced external forces must be balanced by an inertial force in that direction, so that
\[ \rho \, dx \, dy \, dz \frac{\partial^2 u}{\partial t^2} = \left( \sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x} \right) dy \, dz - \sigma_{xx} \, dx \, dz \]
\[ + \left( \sigma_{xy} + \frac{\partial \sigma_{xy}}{\partial y} \right) dy \, dz - \sigma_{xy} \, dx \, dz \]
\[ + \left( \sigma_{xz} + \frac{\partial \sigma_{xz}}{\partial z} \right) dx \, dy - \sigma_{xz} \, dx \, dy \] (2.23)

Which simplifies to
\[ \rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} \quad (2.24.a) \]

Repeating this operation in the \( y \)- and \( z \)-direction gives
\[ \rho \frac{\partial^2 w}{\partial t^2} = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} \quad (2.24.b) \]
\[ \rho \frac{\partial^2 w}{\partial t^2} = \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial z} \quad (2.24.c) \]

(Equations 2.24) represent the three-dimensional equations of motion of an elastic solid. Note that these equations of motion were derived solely on the basis of equilibrium considerations and thus apply to solids of any stress-strain behavior. To express these equations of motion in terms of displacements, it is again necessary to use a stress-strain relationship and a strain-displacement relationship. Using Hooke’s law as developed, the first of the equations of motion (equation 2.24a) can be written in terms of strains as.

\[ \rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x} \left( \lambda \varepsilon + 2\mu \varepsilon_{xx} \right) + \frac{\partial}{\partial y} \left( \mu \varepsilon_{xy} \right) + \frac{\partial}{\partial z} \left( \mu \varepsilon_{xz} \right) \] (2.25)
Substituting the strain-displacement relationships
\[ \varepsilon_{xx} = \frac{\partial u}{\partial x}, \quad \varepsilon_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}, \quad \varepsilon_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \]

\[ \rho \frac{\partial^2 u}{\partial t^2} = (\lambda + \mu) \frac{\partial \bar{\varepsilon}_x}{\partial x} + \mu \nabla^2 \bar{\varepsilon}_x \] (2.26.a)

Where the Laplacian operator \( \nabla^2 \) represents
\[ \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \]

Repeating this process in the \( y \) and \( z \) directions gives
\[ \rho \frac{\partial^2 v}{\partial t^2} = (\lambda + \mu) \frac{\partial \bar{\varepsilon}_y}{\partial y} + \mu \nabla^2 \bar{\varepsilon}_y \] (2.26.b)
\[ \rho \frac{\partial^2 w}{\partial t^2} = (\lambda + \mu) \frac{\partial \bar{\varepsilon}_z}{\partial z} + \mu \nabla^2 \bar{\varepsilon}_z \] (2.26.c)

### 2.2.10 Solutions of the Three-Dimensional Equations of Motion
Together, (equations 2.26) represent the three-dimensional equations of motion for an isotropic, linear, elastic solid. It turns out that these equations can be manipulated to produce two wave equations. Consequently, only two types of waves can travel through such an unbounded solid. The characteristics of each type of wave will be revealed by their respective wave equations.

The solution for the first type of wave can be obtained by differentiating each of (equations 2.26) with respect to \( x, y, \) and \( z \) and adding the results together to give
\[ \rho \left( \frac{\partial^2 \varepsilon_{xx}}{\partial t^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial t^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial t^2} \right) = (\lambda + \mu) \left( \frac{\partial^2 \bar{\varepsilon}_x}{\partial x^2} + \frac{\partial^2 \bar{\varepsilon}_y}{\partial y^2} + \frac{\partial^2 \bar{\varepsilon}_z}{\partial z^2} \right) + \mu \left( \frac{\partial^2 \varepsilon_{xx}}{\partial x^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial y^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial z^2} \right) \]

Or
\[ \rho \frac{\partial^2 \bar{\varepsilon}_x}{\partial t^2} = (\lambda + \mu) \nabla^2 \bar{\varepsilon}_x + \mu \nabla^2 \bar{\varepsilon}_x \] (2.27)

Rearranging yields the wave equation
\[ \frac{\partial^2 \bar{\varepsilon}_x}{\partial t^2} = \frac{\lambda+2\mu}{\rho} \nabla^2 \bar{\varepsilon}_x \] (2.28)

Recalling that \( \bar{\varepsilon}_x \) is the volumetric strain (which describes deformations that involve no shearing or rotation), this wave equation describes an irrotational, or dilatational, wave. It indicates that a dilatational wave will propagate through the body at a velocity
\[ v_p = \sqrt{\frac{\lambda+2\mu}{\rho}} \] (2.29)

This type of wave is commonly known as a \( p \)-wave (or primary wave) and \( v_p \) is referred to as the \( p \)-wave velocity of the material. The general nature of \( p \)-wave
motion was illustrated earlier. Note that particle displacements are parallel to the direction of wave propagation, just as they were in the constrained rod. The longitudinal wave in the constrained rod is actually a p-wave. Using (equations 2.22c and 2.22d), \( v_p \) can be written in terms of the shear modulus and Poisson’s ratio as

\[
v_p = \sqrt{\frac{G(2-2\nu)}{\rho(1-2\nu)}} \tag{2.30}
\]

As \( \nu \) approaches 0.5 (which point the body becomes incompressible, i.e., infinitely stiff with respect to dilatational deformations), \( v_p \) approaches infinity.

To obtain the solution for the second type of wave, \( \ddot{\varepsilon} \) is eliminated by differentiating (equation 2.26.b) with respect to \( z \) and (equation 2.26c) with respect to \( y \), and subtracting one from the other:

\[
\rho \frac{\partial}{\partial t^2} \left( \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) = \mu \nabla^2 \left( \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \tag{2.31}
\]

Recalling the definition of rotation (equation 2.19), (equation 2.31) can be written in the form of the wave equation which describes an equivoluminal, or distortional wave, of rotation about the \( x \)-axis.

\[
\frac{\partial^2 \Omega_x}{\partial t^2} = \frac{\mu}{\rho} \nabla^2 \Omega_x \tag{2.32}
\]

Similar expressions can be obtained by the same process for rotation about the \( y \)- and \( z \)-axes. Equation 32 shows that a distortional wave will propagate through the solid at a velocity

\[
v_s = \sqrt{\frac{\mu}{\rho}} = \sqrt{\frac{G}{\rho}} \tag{2.33}
\]

This type of wave is commonly known as a s-wave (or shear wave) and \( v_s \) is referred to as the shear wave velocity of the material. Note that the particle motion is constrained to a plane perpendicular to the direction of wave propagation, just as it was in the case of the torsional wave. Consequently, the torsional wave represented a form of an s-wave. The close relationship between s-wave velocity and shear modulus is used to advantage in many of the field and laboratory tests discussed in module 6. The general nature of s-wave motion was illustrated.

S-waves are often divided into two types, or resolved into two perpendicular components. SH-waves in which particle motion occurs only in a horizontal plane. SV-waves are s-waves whose particle motion lies in a vertical plane. A given s-wave with arbitrary particle motion can be represented as the vector sum of its SH and SV components.

In summary, only two types of waves, known as body waves, can exist in an
unbounded (infinite) elastic solid. P-waves involve no rotation of the material they pass through and travel at velocity, $v_p$. S-waves involve no volume change and travel at velocity, $v_s$. The velocity of p- and s-waves depend on the stiffnesses of the solid with respect to the types of deformation induced by each wave. Comparing the velocities (equation 2.30 and 2.33) the p-wave velocity can be seen to exceed the s-wave velocity by an amount that depends on the compressibility (as reflected in Poisson’s ratio) of the body.

$$\frac{v_p}{v_s} = \sqrt{\frac{2-2\nu}{1-2\nu}}$$  \hspace{1cm} (2.34)

For a typical Poisson’s ratio of 0.3 for geologic materials, the ratio $v_p/v_s = 1.87$. 