Module 6

Lecture 3: Derivation of momentum equation
Velocity Distribution in an Open Channel Flow

Diagrammatic representation of (a) Isovels and (b) Velocity profile in an open channel flow
Gradually Varied Flow

Hydrostatic pressure distribution in a curved surface is,

Convex Curvi-linear flow

For horizontal surface

For curved surface

\[ \gamma h \]

\[ \gamma(h-a_n h/g) \]

\[ \gamma a_n h/g \]

\[ y \]

\[ a_n \text{: acceleration component} \]

acting normal to the streamlines

Convex surface

Module 6
Gradually Varied Flow

For horizontal surface

For curved surface

Concave Curvi-linear flow

Concave surface

\[ \gamma h \]

\[ \gamma a_n h/g \]

\[ y \]

\[ a_n \]

\[ h \]
Note: \( y = y(x,t) \), Depth of flow varies with distance and time
\( A = A(x,y) \), Area of flow

Information can be known about:
1) Critical inflow hydrograph
2) How much area may be flooded for critical cases

From Reynold’s transport theorem, we have

For mass \((m)\) \(\Rightarrow\) \( \beta = \frac{d(m)}{dm} = 1 \)
Consider $B$ to be mass.

Now, as per law of conservation of mass,

$$\frac{dB_{\text{sys}}}{dt} = \frac{d(\text{mass})}{dt} = 0$$

This is the equation of variable density unsteady flow, and $\rho$ is constant.

$$\Rightarrow \frac{\partial}{\partial t} \iiint \rho d\mathcal{V} + \iint \rho \vec{V} \cdot d\mathbf{A} = 0$$

$$\frac{\partial}{\partial t} \text{ indicates that all the variables are function of } x \text{ & } t$$
or \[ 0 = \frac{\partial (\rho \cdot A \cdot dx)}{\partial t} + \iiint_{\text{inlet}} \rho \bar{V} \cdot dA + \iiint_{\text{outlet}} \rho \bar{V} \cdot dA \]  

(6.21)
Component in $x$-direction

$q/2$

$dx$

Top view of the open channel section

Total contribution is $q \cdot dx$

Cross-sectional view of a compound channel

Module 6
Equation (6.21) can be re-written as,

\[ 0 = \frac{\partial (\rho A dx)}{\partial t} - \rho (Q + q dx) + \rho \left( Q + \left( \frac{\partial Q}{\partial x} \right) dx \right) \]

\[ = \frac{\partial (\rho A dx)}{\partial t} - \rho q dx + \rho \left( \frac{\partial Q}{\partial x} \right) dx \]

\[ = \frac{\partial (\rho A dx)}{\partial t} - \rho \left( q dx - \left( \frac{\partial Q}{\partial x} \right) dx \right), \]

Dividing both sides by \( \rho dx \),

\[ 0 = \frac{\partial A}{\partial t} - \left( q - \frac{\partial Q}{\partial x} \right) \]

or

\[ q = \frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} \]

Basic equation or Conservative form of continuity equation (Applicable for kinematic & non-prismatic channels)
Now, \( q = \frac{\partial A}{\partial t} + \frac{\partial (A \cdot V)}{\partial x} \)

or \( q = \frac{\partial A}{\partial t} + A \frac{\partial V}{\partial x} + V \frac{\partial A}{\partial x} \)

Here \( A = f(y, t); y = y(x, t) \)

or \( q = \begin{bmatrix} \frac{\partial A}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial y}{\partial t} \end{bmatrix} + A \begin{bmatrix} \frac{\partial V}{\partial x} \end{bmatrix} + V \begin{bmatrix} \frac{\partial A}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial y}{\partial x} \end{bmatrix} \)

For prismatic channel, \( \frac{\partial A}{\partial x} = 0 \)
\[
dA = B' \, dy
\]

So, \( \frac{\partial A}{\partial y} \approx B' \)

or \( q = B' \left[ \frac{\partial y}{\partial t} \right] + A \left[ \frac{\partial V}{\partial x} \right] + V \left[ 0 + B' \frac{\partial y}{\partial x} \right] \), for prismatic channel

Now, hydraulic depth \( (D) = A / B' \Rightarrow A = B'.D \)

Here \( A = f(y,t); \, y = y(x,t) \)

or \( q = B' \left[ \frac{\partial y}{\partial t} \right] + B'.D \left[ \frac{\partial V}{\partial x} \right] + V.B' \left[ \frac{\partial y}{\partial x} \right] \)

Assume \( g, \quad q = 0 \) & \( B' = 1 \) (unit width of the channel)

or \( 0 = \frac{\partial y}{\partial t} + D \left[ \frac{\partial V}{\partial x} \right] + V \left[ \frac{\partial y}{\partial x} \right] \)
If depth is same (uniform) throughout the channel length,
i.e., \( D = \frac{A}{B'} = B'.y / B' = y \)

or

\[
O = \frac{\partial y}{\partial t} + y \left[ \frac{\partial V}{\partial x} \right] + V \left[ \frac{\partial y}{\partial x} \right]
\]

Now, from Reynolds transport theorem:

\[
\frac{dB}{dt} = \frac{\partial}{\partial t} \int \int \int \int \beta \rho dA + \int \int \beta \rho \vec{V} \cdot dA
\]

From conservation law of momentum
(Newton’s 2\textsuperscript{nd} law of motion)

\[
\frac{dB}{dt} = \sum F
\]

momentum \( (m \vec{v}) \) \( \Rightarrow \) \( \beta = \frac{d(m \vec{v})}{dm} = \vec{v} \)
\[
\sum F = \frac{\partial}{\partial t} \iiint_V \rho d\mathbf{v} + \iint_S \mathbf{V} \cdot \rho \mathbf{V} \cdot d\mathbf{A}
\]

For steady uniform flow, \( \sum F = 0 \)
Forces acting on the C.V.

- Elevation View
- Plan View

Module 6
**Forces acting on the C.V.**

- \( F_g \) = **Gravity force** due to weight of water in the C.V.
- \( F_f \) = **friction force** due to shear stress along the bottom and sides of the C.V.
- \( F_e \) = **contraction/expansion force** due to abrupt changes in the channel cross-section
- \( F_w \) = **wind shear force** due to frictional resistance of wind at the water surface
- \( F_p \) = **unbalanced pressure forces due to hydrostatic forces** on the left and right hand side of the C.V. and pressure force exerted by banks
a) Hydrostatic Force

Area of the Elemental strip = \( b \cdot d \eta \)

Force = area * hydrostatic pressure

\[
F_y = \int_{\eta=0}^{y} (b \cdot d\eta) \gamma (y - \eta) \, d\eta
\]

or \( dF = (b \cdot d\eta) \gamma (y - \eta) \)
a) Hydrostatic Force

\[ \&F_2 = F_1 + \left( \frac{\partial F_1}{\partial x} \right) dx \]

\[ \&F_p = (F_1 - F_2) \]

\[ = F_1 - \left( F_1 + \left( \frac{\partial F_1}{\partial x} \right) dx \right) = - \left( \frac{\partial F_1}{\partial x} \right) dx \]

\[ \Rightarrow F_p = - \left[ \frac{\partial}{\partial x} \int_0^{y=\eta} (b \cdot d\eta) \gamma (y - \eta) \right] dx \]

and \( y = y(x, t) \) (using Leibnitz rule)

**General rule:**

\[ F(t) = \int_{a(t)}^{b(t)} \phi(x, t) dx \]

\[ \frac{d}{dt} \{ F(t) \} = \frac{d}{dt} \left[ \int_{a(t)}^{b(t)} \phi(x, t) dx \right] \]
\[= \left[ \int_{a(t)}^{b(t)} \left( \frac{d\phi(x,t)}{dt} \right) dx \right] + \left\{ \phi(b(t),t) \cdot \frac{db(t)}{dt} \right\} - \left\{ \phi(a(t),t) \cdot \frac{da(t)}{dt} \right\} \]

Therefore,

\[F_p = -\left[ \int_{\eta=0}^{\eta=y} \frac{\partial}{\partial x} \left( b \gamma (y - \eta) \right) d\eta \right] - \left[ b \gamma (y - y) \frac{d(\eta)}{dx} \right] + \left[ b \gamma (y - 0) \frac{d(0)}{dx} \right] \]

\[= -\left[ \int_{\eta=0}^{\eta=y} \gamma \left[ b \frac{\partial}{\partial x} (y - \eta) + (y - \eta) \frac{\partial(b)}{\partial x} \right] d\eta \right] dx \]

\[= -\left[ \int_{0}^{\gamma} \left\{ \gamma b \left( \frac{\partial y}{\partial x} - \frac{\partial \eta}{\partial x} \right) + \gamma (y - \eta) \frac{\partial b}{\partial x} \right\} d\eta \right] dx \]

\[= \left[ \int_{0}^{\gamma} \gamma b \frac{\partial y}{\partial x} d\eta + \int_{0}^{\gamma} \gamma (y - \eta) \frac{\partial b}{\partial x} d\eta \right] dx \]

\[= -\left[ \gamma \frac{\partial y}{\partial x} \int_{0}^{\gamma} b d\eta \right] dx - \left[ \int_{0}^{\gamma} \gamma (y - \eta) \frac{\partial b}{\partial x} d\eta \right] dx \]

\[= -\left[ \gamma \frac{\partial y}{\partial x} A \right] dx - \left[ \int_{0}^{\gamma} \gamma (y - \eta) \frac{\partial b}{\partial x} d\eta \right] dx \]
b) Boundary Reaction

Elementary area = \( b.d\eta = A_{\text{element at section}} \) (1) – (1)

\[
\left( \frac{\partial A_{\text{element}}}{\partial x} \right) dx + A_{\text{element}} = \text{Elemental area at section} \) (2) – (2)

∴ Change in elemental area,

\[
= \left( \frac{\partial A_{\text{element}}}{\partial x} \right) dx = \left( \frac{\partial (b.d\eta)}{\partial x} \right) dx = \frac{\partial b}{\partial x} d\eta dx
\]

∴ Change in the hydrostatic pressure distribution also.
In similar way,
\[ \int_0^y \frac{\partial}{\partial x} (bd\eta) dx \cdot \gamma (y - \eta), \]
additional hydrostatic force for change in width of the channel.
c) Body Force

Let area \((A)\) be constant,

\[
\therefore \text{Weight}, \ W = A \gamma \, dx
\]

\[
\therefore F_g = W \cdot \sin \theta
\]
c) Body Force

For very small values of $\theta$, $\sin \theta \cong \tan \theta$ and $\tan \theta = S_0$

or $F_g = \gamma . A . dx . S_0$

Shear force for channel bottom and sides is,

$P . dx . \tau_0 = \gamma . A . dx . S_0$ where $P = \text{wetted perimeter}$, $\tau_0 = \text{shear stress}$

(.: Assumptions: These two forces are entirely balanced)

or $\tau_0 = \gamma \left( \frac{A}{P} \right) S_0 = \gamma . R . S_0$

where $R = \text{hydraulic radius}$ or $\text{hydraulic mean depth}$

For steady uniform flow, $S_0 = S_f$

However, we need to consider unsteady uniform flow here.
d) Frictional Force

\[ \tau_0 = \gamma R S_f \]
\[ \therefore \text{ Shear force} = -Pdx\tau_0 \]
\[ = -Pdx(\gamma R S_f) \]
\[ \text{or} \quad F_f = -\gamma A dx S_f \]

e) Wind Force

Let wind shear stress be \( \tau_w \)
\[ F_w = \text{windforce} = B dx \tau_w \quad \text{(If top width is B and it is uniform throughout the length of the channel)} \]
\[ \tau_w = -\frac{\rho C_f |V_r| V_r}{2} = -\rho W_f \]
If } V_r \text{ is } +ve \Rightarrow |V_r| \text{ is } +ve, \text{ so } \tau_w \text{ is } -ve.
If } V_r \text{ is } -ve \Rightarrow |V_r| \text{ is } -ve, \text{ so } \tau_w \text{ is } +ve.

C_f = \text{ wind shear stress coefficient}
W_f = \text{ wind shear stress factor}
\therefore F_w = B.dx(-\rho W_f) = -B.dx.\rho.W_f

f) Eddy Losses:

Eddy loss \Rightarrow \text{ Slope of the energy gradient line}(S_e)

S_e \propto \frac{\partial}{\partial x} \left( \frac{v^2}{2g} \right)

or \quad S_e = K_e \frac{\partial}{\partial x} \left( \frac{v^2}{2g} \right)

S_e = \frac{K_e}{2g} \frac{\partial}{\partial x} (v^2) = \frac{K_e}{2g} \frac{\partial}{\partial x} \left( \frac{Q}{A} \right)^2
Drag force causing the eddy loss: \( F_e = -\gamma A dx S_e \)

Here \( K_e = \text{contraction or expansion coefficient} \)

Obviously \( V_2 \gg V_1 \)

Ensure that \( K_e \) is always (+ve) in all the cases.

Now, momentum equation,

\[
\sum F = \frac{d}{dt} \iiint V \cdot \rho \, dA \, \text{cv} + \iiint V \cdot \rho \vec{V} \cdot d\vec{A} \]

as \( \beta = \frac{d \left( \rho \vec{V} \right)}{dm} = \vec{v} \)

\[
\sum F = \frac{d}{dt} \left( \rho \vec{V} \cdot A \, dx \right) + \iiint V \cdot \rho \vec{V} \cdot d\vec{A} \, \text{inlet} + \iiint V \cdot \rho \vec{V} \cdot d\vec{A} \, \text{outlet}
\]
Forces acting on the C.V. (Contd.)

\[ \beta_1(\rho\bar{V}.Q) \]

\[ \beta_1(\rho\bar{V}.Q + \frac{\partial}{\partial x}(\rho\bar{V}.Q)dx) \]

\[ \sum F = \frac{d}{dt}(\rho\bar{V}.\bar{A}.dx) + \int_{\text{inlet}} V . \rho \bar{V} . d\bar{A} + \int_{\text{outlet}} V . \rho \bar{V} . d\bar{A} \]  

(1)

Here \( \beta_1 \) = momentum correction factor

Consider \( V_x \) to be component of velocity along (+ve) direction

and \( Q = q.dx, \ \beta_1.\rho\bar{V}.Q = \beta_1.\rho.V_x.(q.dx) \)
Now, equation (1),

\[
\frac{d}{dt} (\rho V \cdot A \cdot dx) - (\beta_1 \rho V \cdot Q + \beta_1 \rho V_x \cdot (q \cdot dx)) + \beta_1 \left( \rho V \cdot Q + \frac{\partial}{\partial x} (\rho V \cdot Q) \cdot dx \right)
\]

\[= \left[ \frac{\partial}{\partial t} (V \cdot \rho \cdot A \cdot dx) - \beta_1 \left( \rho V_x \cdot q \cdot dx + \frac{\partial}{\partial x} (\rho V \cdot Q) \cdot dx \right) \right] \]

Now, combining all the expansions to find out the total force:

\[
\rho g A dx S_0 - \rho g A dx S_f - \rho g A dx S_e - \rho W_f B dx - \gamma \frac{\partial y}{\partial x} A dx - \int_0^y \gamma (y - \eta) \frac{\partial b}{\partial x} \, d\eta \, dx
\]

Now, equation (1),

\[
\frac{d}{dt} (\rho V \cdot A \cdot dx) - (\beta_1 \rho V \cdot Q + \beta_1 \rho V_x \cdot (q \cdot dx)) + \beta_1 \left( \rho V \cdot Q + \frac{\partial}{\partial x} (\rho V \cdot Q) \cdot dx \right)
\]

\[= \left[ \frac{\partial}{\partial t} (V \cdot \rho \cdot A \cdot dx) - \beta_1 \left( \rho V_x \cdot q \cdot dx + \frac{\partial}{\partial x} (\rho V \cdot Q) \cdot dx \right) \right] \]

Now, combining all the expansions to find out the total force:

\[
\rho g A dx S_0 - \rho g A dx S_f - \rho g A dx S_e - \rho W_f B dx - \gamma \frac{\partial y}{\partial x} A dx - \int_0^y \gamma (y - \eta) \frac{\partial b}{\partial x} \, d\eta \, dx
\]
Now, dividing L.H.S & R.H.S by "ρdx":

\[
\begin{align*}
\text{or} & \quad \left[ gAS_0 - gAS_f - gAS_e - W_f B - g \frac{\partial y}{\partial x} A \right] \\
& = \left[ \frac{\partial}{\partial t} (V \cdot A) - \beta_1 \left( V_x \cdot q + \frac{\partial}{\partial x} (V \cdot Q) dx \right) \right] \\
& = \left[ \frac{\partial Q}{\partial t} - \beta_1 V_x \cdot q + \beta_1 \frac{\partial}{\partial x} (V \cdot Q) \right] \\
& = \left[ \frac{\partial Q}{\partial t} - \beta_1 V_x \cdot q + \beta_1 \frac{\partial}{\partial x} \left( \frac{Q^2}{A} \right) \right]
\end{align*}
\]

Assume \( W_f = 0 \) & \( S_e = 0 \),

\[
gA(S_0 - S_f) - gA \frac{\partial y}{\partial x} = \frac{\partial Q}{\partial t} - \beta_1 V_x \cdot q + \beta_1 \frac{\partial}{\partial x} \left( \frac{Q^2}{A} \right)
\]

Now, dividing L.H.S & R.H.S by "gA":

\[
\text{or} \quad \left( S_0 - S_f \right) - \frac{\partial y}{\partial x} = \frac{1}{gA} \left( \frac{\partial Q}{\partial t} \right) - \frac{\beta_1 V_x \cdot q}{gA} + \frac{\beta_1}{gA} \frac{\partial}{\partial x} \left( \frac{Q^2}{A} \right)
\]
Conservation form of momentum equation (for prismatic channels):

If \( q = 0 \) and \( \beta_1 = 1 \),

\[
\frac{1}{gA} \frac{\partial Q}{\partial t} + \frac{1}{gA} \frac{\partial}{\partial x} \left( \frac{Q^2}{A} \right) - \frac{\partial y}{\partial x} - (S_o - S_f) = 0
\]

Multiplying L.H.S & R.H.S by "g",

\[
\frac{1}{A} \frac{\partial Q}{\partial t} + \frac{1}{A} \frac{\partial}{\partial x} \left( \frac{Q^2}{A} \right) + g \frac{\partial y}{\partial x} - g(S_o - S_f) = 0
\]
Conservation form of Momentum Equation:

\[
\frac{1}{A} \frac{\partial Q}{\partial t} + \frac{1}{A} \frac{\partial}{\partial x} \left( \frac{Q^2}{A} \right) + g \frac{\partial y}{\partial x} - g(S_o - S_f) = 0
\]

Taking, \( Q = A.V \)

or \[
\frac{1}{A} \frac{\partial (A.V)}{\partial t} + \frac{1}{A} \frac{\partial}{\partial x} \left( \frac{A^2.V^2}{A} \right) + g \frac{\partial y}{\partial x} - g(S_o - S_f) = 0
\]

or \[
\frac{1}{A} \left[ A \frac{\partial V}{\partial t} + V \frac{\partial A}{\partial t} \right] + \frac{1}{A} \left[ 2A.V \frac{\partial V}{\partial t} \right] + \frac{1}{A} \left[ V^2 \frac{\partial A}{\partial t} \right] + \frac{1}{A} \left[ V^2 \frac{\partial A}{\partial y} \frac{\partial y}{\partial x} \right] + g \frac{\partial y}{\partial x} - g(S_o - S_f) = 0
\]

or \[
\frac{\partial V}{\partial t} + V \left[ \frac{\partial A}{\partial y} \frac{\partial y}{\partial t} \right] + 2V \left[ \frac{\partial V}{\partial y} \right] + \frac{V^2}{A} \left[ \frac{\partial A}{\partial x} \right] + \frac{\partial A}{\partial y} \frac{\partial y}{\partial x} + g \frac{\partial y}{\partial x} - g(S_o - S_f) = 0
\]

Now, for prismatic channel, \( \left. \frac{\partial A}{\partial x} \right|_{y = \text{const} \ tan t} = 0 \)

or \[
\frac{\partial V}{\partial t} + V \left[ \frac{\partial A}{\partial y} \frac{\partial y}{\partial t} \right] + 2V \left[ \frac{\partial V}{\partial y} \right] + \frac{V^2}{A} \left[ \frac{\partial A}{\partial y} \cdot \frac{\partial y}{\partial x} \right] + g \frac{\partial y}{\partial x} - g(S_o - S_f) = 0
\]
Continuity Equation for Non-Conservation form (=0)

\[
\frac{\partial V}{\partial t} + \frac{V}{A} \left[ B \frac{\partial y}{\partial t} \right] + V \left[ \frac{\partial V}{\partial x} \right] + V \left[ \frac{\partial V}{\partial x} \right] + \frac{V^2}{A} \left[ \frac{\partial A}{\partial y} \frac{\partial y}{\partial x} \right] + g \frac{\partial y}{\partial x} - g(S_o - S_f) = 0
\]
\[-\frac{1}{g} \frac{\partial V}{\partial t} - \frac{V}{g} \frac{\partial V}{\partial x} - \frac{\partial y}{\partial x} + S_o = S_f\]

**Steady, uniform flow**

**Steady, non-uniform flow**

**Unsteady, non-uniform flow**

But if there is internal flow,

\[\frac{\partial V}{\partial t} + V \frac{\partial V}{\partial x} + g \frac{\partial y}{\partial x} - g(S_o - S_f) = \frac{\beta_1 V_x q}{A}\]

Dividing L.H.S & R.H.S by "g",

\[\frac{1}{g} \frac{\partial V}{\partial t} + \frac{V}{g} \frac{\partial V}{\partial x} + \frac{\partial y}{\partial x} - (S_o - S_f) = \frac{\beta_1 V_x q}{gA}\]
Here \( \frac{\partial V}{\partial t} \Rightarrow \text{local acceleration term (for steady flow = 0)} \)

\( \frac{\partial V}{\partial x} \Rightarrow \text{convective acceleration term (for uniform flow = 0)} \)

**For steady – uniform flow,**

\[-(S_o - S_f) = \frac{\beta_1 V_x q}{gA}\]

or \( (S_o - S_f) + \frac{\beta_1 V_x q}{gA} = 0 \) \hspace{1cm} (6.22)

**For steady non-uniform flow,**

\[\frac{V}{g} \frac{\partial V}{\partial x} + \frac{\partial y}{\partial x} - (S_o - S_f) = \frac{\beta_1 V_x q}{gA}\] \hspace{1cm} (6.23)

**For unsteady non-uniform flow,**

\[\frac{1}{g} \frac{\partial V}{\partial t} + \frac{V}{g} \frac{\partial V}{\partial x} + \frac{\partial y}{\partial x} - (S_o - S_f) = \frac{\beta_1 V_x q}{gA}\] \hspace{1cm} (6.24)