Advanced Numerical Analysis for Chemical Engineering
Mid-Term Examination -1 (2 hrs.)

1. Let $X$ represent set of continuous functions on interval $0 \leq t \leq 1$ with inner product defined as

$$\langle x(t), y(t) \rangle = \int_0^1 w(t)x(t)y(t)\,dt$$

Given a set of two linearly independent vectors

$$x^{(1)}(t) = 1; \quad x^{(2)}(t) = t$$

find orthonormal set of vectors $e^{(1)}(t)$ and $e^{(2)}(t)$ if $w(t) = t(1-t)$. (6 marks)

2. Consider system $Ax = b$ where

$$A = \begin{bmatrix} 3 & 1 & -2 \\ 3 & 1 & -2 \\ 3 & 1 & -2 \end{bmatrix}$$

1. What is the dimension of column space of matrix $A$, i.e. $R(A)$? Find a basis for $R(A)$. (6 marks)

2. What is the dimension of null space of matrix $A$, i.e. $N(A)$? Find a basis for $R(A)$. (6 marks)

   Hint: Note that, for a $n \times n$ matrix $A$

   $$\dim[R(A)] = \dim[R(A^T)]$$
   $$\dim[R(A^T)] + \dim[N(A)] = n$$

3. It is proposed to define an inner product as follows

$$\langle x, y \rangle = x^T Ay$$

Is this a valid definition of inner product? Justify your answer. (4 marks)
3. Application of finite difference method to solve a PDE resulted in a set of linear algebraic equation, $Ax = b$, where

$$A = \begin{bmatrix}
4 & -1 & 0 & -1 & 0 & 0 \\
-1 & 4 & -1 & 0 & -1 & 0 \\
0 & -1 & 4 & 0 & 0 & -1 \\
-1 & 0 & 0 & 4 & -1 & 0 \\
0 & -1 & 0 & -1 & 4 & -1 \\
0 & 0 & -1 & 0 & -1 & 4 \\
\end{bmatrix}; \quad b = \begin{bmatrix}
2 \\
1 \\
2 \\
2 \\
1 \\
2 \\
\end{bmatrix}$$

It is desired to solve the above equation using Jacobi and Gauss-Seidel methods starting from the following initial guess solution

$$x^{(0)} = \begin{bmatrix}
-5 \\
2 \\
-3 \\
1 \\
-3 \\
5 \\
\end{bmatrix}^T$$

Will the iterations converge in each case? Justify your answer. (4 marks)

4. Consider the following difference equation initial value problem

$$e^{(k+1)} = Ae^{(k)}; \quad A = \begin{bmatrix}
-2 & 1 \\
1 & -2 \\
\end{bmatrix}; \quad e^{(0)} = \begin{bmatrix}
1 \\
1 \\
\end{bmatrix}$$

1. Find eigen-values and eigen vectors of $A$. (3 marks)

2. Comment upon the asymptotic behavior of the solution $e^{(k)}$ for large $k$ (i.e. $k \to \infty$). (3 marks)