Module 2: Fundamentals of Vector Spaces

Section 3: Normed Linear Spaces and Banach Spaces

3. Normed Linear Spaces and Banach Spaces

In three dimensional space, we use *lengths or magnitudes* to compare any two vectors. Generalization of the concept of length / magnitude of a vector in three dimensional vector space to an arbitrary vector space is achieved by defining a scalar valued function called *norm* of a vector.

**Definition 28 (Normed Linear Vector Space):** A normed linear vector space is a vector space $X$ on which there is defined a real valued function which maps each element $x \in X$ into a real number $\| x \|$ called norm of $x$. The norm satisfies the following axioms.

1. $\| x \| \geq 0$ for all $x \in X$; $\| x \| = 0$ if and only if $x = \overline{0}$ (zero vector)
2. $\| x + y \| \leq \| x \| + \| y \|$ for each $x, y \in X$. (triangle inequality).
3. $\| \alpha x \| = |\alpha| \cdot \| x \|$ for all scalars $\alpha$ and each $x \in X$

**Example 29 Vector norms:**

1. $(\mathbb{R}^n, \| \cdot \|_1)$ : Euclidean space $\mathbb{R}^n$ with 1-norm: $\| x \|_1 = \sum_{i=1}^{N} |x_i|$

2. $(\mathbb{R}^n, \| \cdot \|_2)$ : Euclidean space $\mathbb{R}^n$ with 2-norm:

   $$\| x \|_2 = \left[ \sum_{i=1}^{N} (x_i)^2 \right]^{1/2}$$

3. $(\mathbb{R}^n, \| \cdot \|_p)$ : Euclidean space $\mathbb{R}^n$ with $p$-norm:

   $$\| x \|_p = \left[ \sum_{i=1}^{N} |x_i|^p \right]^{1/p} \quad \text{(13)}$$

   where $p$ is a positive integer

4. $(\mathbb{R}^n, \| \cdot \|_\infty)$ : Euclidean space $\mathbb{R}^n$ with $\infty$-norm: $\| x \|_\infty = \max |x_i|$

5. $n$-dimensional complex space $(\mathbb{C}^n)$ with $p$-norm:

   $$\| x \|_p = \left[ \sum_{i=1}^{N} |x_i|^p \right]^{1/p} \quad \text{(14)}$$

   where $p$ is a positive integer

6. Space of infinite sequences $(l_\infty)$ with $p$-norm: An element in this space, say $x \in l_\infty$, is an infinite sequence of numbers $x = \{x_1, x_2, \ldots, x_n, \ldots\}$ such that $p$-norm defined as:

   $$\| x \|_p = \left[ \sum_{i=1}^{N} |x_i|^p \right]^{1/p} \quad \text{(16)}$$
\[ \| x \|_p = \left( \sum_{i=1}^{\infty} |x_i|^p \right)^{\frac{1}{p}} < \infty \]

is bounded for every \( x \in l_{\infty} \), where \( p \) is an integer.

7. \((C[a,b], \| x(t) \|_\infty)\) : The normed linear space \( C[a,b] \) together with infinite norm

\[ \| x(t) \|_\infty = \max_{a \leq t \leq b} |x(t)| \tag{17} \]

It is easy to see that \( \| x(t) \|_\infty \) defined above qualifies to be a norm

\[ \max |x(t) + y(t)| \leq \max |x(t)| + |y(t)| \leq \max |x(t)| + \max |y(t)| \tag{18} \]

\[ \max |\alpha x(t)| = |\alpha| \max |x(t)| \tag{19} \]

8. Other types of norms, which can be defined on the set of continuous functions over \([a, b]\) are as follows

\[ \| x(t) \|_1 = \int_{a}^{b} |x(t)| dt \tag{20} \]

\[ \| x(t) \|_2 = \left( \int_{a}^{b} |x(t)|^2 dt \right)^{\frac{1}{2}} \tag{21} \]

**Example 30** Determine whether (a) \( \max |df(t)| dt \) (b) \( \max |x(t)| + \max |x'(t)| \) (c) \( |x(a)| + \max |x'(t)| \) and (d) \( |x(a)| \max |x(t)| \) can serve as a valid definitions for norm in \( C^{(2)}[a, b] \).

**Solution:** (a) \( \max |df(t)| dt \) : For this to be a norm function, Axiom 1 in the definition of the normed vector spaces requires

\[ \| f(t) \| = 0 \Rightarrow f(t) \text{ is the zero vector in } C^{(2)}[a,b] \text{ i.e. } f(t) = 0 \text{ for all } t \in [a,b] \]

However, consider the constant function i.e. \( g(t) = c \) for all \( t \in [a,b] \) where \( c \) is some non-zero value. It is easy to see that

\[ \max |dg(t)| dt = 0 \]

even when \( g(t) \) does not correspond to the zero vector. Thus, the above function violates Axiom 1 in the definition of a normed vector space and, consequently, cannot qualify as a norm.

(b) \( \max |x(t)| + \max |x'(t)| \) : For any non-zero function \( x(t) \in C^{(2)}[a,b] \), Axiom 1 is satisfied. Axiom 2 follows from the following inequality

\[ \| x(t) + y(t) \| = \max |x(t) + y(t)| + \max |x'(t) + y'(t)| \]

\[ \leq [\max |x(t)| + \max |y(t)|] + [\max |x'(t)| + \max |y'(t)|] \]

\[ \leq [\max |x(t)| + \max |x'(t)|] + [\max |y(t)| + \max |y'(t)|] \]

\[ \leq \| x(t) \| + \| y(t) \| \]

It is easy to show that Axiom 3 is also satisfied for all scalars \( \alpha \). Thus, given function defines a norm on
$C^{(2)}[a, b]$

(c) $|x(a)| + \max|x'(t)|$: For any non-zero function $x(t) \in C^{(2)}[a, b]$, Axiom 1 is satisfied. Axiom 2 follows from the following inequality
\[
\|x(t) + y(t)\| = |x(a) + y(a)| + \max|x'(t)| + y'(t)| \\
\leq |x(a)| + |y(a)| + [\max|x'(t)| + \max|y'(t)|] \\
\leq |x(a)| + \max|x'(t)| + |y(a)| + \max|y'(t)| \\
\leq \|x(t)\| + \|y(t)\|
\]

Axiom A3 is also satisfied for any $\alpha$ as
\[
\|\alpha x(t)\| = |\alpha x(a)| + \max|\alpha x'(t)| \\
= |\alpha| [\max|x(a)| + \max|x'(t)|] \\
= |\alpha| \|x\|
\]

(d) $|x(a)|\max|x(t)|$: Consider a non-zero function $x(t)$ in $C^{(2)}[a, b]$ such that $x(a) = 0$ and $\max|x(t)| \neq 0$. Then, Axiom 1 is not satisfied for all vector $x(t) \in C^{(2)}[a, b]$ and the above function does not qualify to be a norm on

In a normed linear space $X$, the set of all vectors $x \in X$ such that $\|x - \overline{x}\| \leq 1$ is called unit ball centered at $\overline{x}$. A unit ball in $(\mathbb{R}^2, \|\cdot\|_2)$ is the set of all vectors in the circle with the origin at the center and radius equal to one while a unit ball in $(\mathbb{R}^3, \|\cdot\|_2)$ is the set of all points in the unit sphere with the origin at the center. Schematic representation of a unit ball in $C[0,1]$ when maximum norm is used is shown in Figure 1. The unit ball in $C[0,1]$ is set of all functions $f(z)$ such that $|f(z)| \leq 1$ where $z \in [0, 1]$.

![Figure 1: Representation of a unit ball in C[0,1]](image)

Once we have defined a norm in a vector space, we can proceed to generalize the concept of convergence of a sequence of vector. Concept of convergence is central to all iterative numerical methods.

**Definition 31 (Cauchy sequence)**: A sequence $\{x^{(n)}\}$ in normed linear space is said to be a Cauchy sequence if $\|x^{(n)} - x^{(m)}\| \to 0$ as $n, m \to \infty$, i.e. given an $\varepsilon > 0$ there exists an integer $N$ such that $\|x^{(n)} - x^{(m)}\| < \varepsilon$ for all $n, m \geq N$.

**Definition 32 (Convergence)**: In a normed linear space an infinite sequence of vectors $\{x^{(k)} : k = 1, 2, \ldots\}$ is said to converge to a vector $x^*$ if the sequence
\[ \{ \| x^k - x^{(k)} \|, \; k = 1, 2, \ldots \} \] of real numbers converges to zero. In this case we write \( x^{(k)} \to x^* \).

In particular, a sequence \( \{ x^{(k)} \} \) in \( \mathbb{R}^n \) converges if and only if each component of the vector sequence converges. If a sequence converges, then its limit is unique.

**Example 33 Convergent sequences:** Consider the sequence of vectors represented as

\[
x^{(k)} = \begin{bmatrix}
1 + (0.2)^k \\
-1 + (0.9)^k \\
3/(1 + (-0.5)^k) \\
(0.8)^k
\end{bmatrix} \to \begin{bmatrix}
1 \\
-1 \\
3 \\
0
\end{bmatrix}
\]

for \( k = 0, 1, 2, \ldots \) is a convergent sequence with respect to any p-norm defined on \( \mathbb{R}^4 \). It can be shown that it is a Cauchy sequence. Note that each element of the vector converges to a limit in this case.

Every convergent sequence is a Cauchy sequence. Moreover, when we are working in \( \mathbb{R}^n \) or \( \mathbb{C}^n \), all Cauchy sequences are convergent. However, all Cauchy sequences in a general vector space need not be convergent. Cauchy sequences in some vector spaces exhibit such strange behavior and this motivates the concept of completeness of a vector space.

**Definition 34 (Banach Space):** A normed linear space \( X \) is said to be complete if every Cauchy sequence has a limit in \( X \). A complete normed linear space is called Banach space.

Examples of Banach spaces are

\[
(\mathbb{R}^n, \| \cdot \|_1), (\mathbb{R}^n, \| \cdot \|_2), (\mathbb{R}^n, \| \cdot \|_{\infty})
\]

\[
(\mathbb{C}^n, \| \cdot \|_1), (\mathbb{C}^n, \| \cdot \|_2), (l_\infty, \| \cdot \|_1), (l_\infty, \| \cdot \|_2)
\]

Concept of Banach spaces can be better understood if we consider an example of a vector space where a Cauchy sequence is not convergent, i.e. the space under consideration is an incomplete normed linear space. Note that, even if we find one Cauchy sequence in this space which does not converge, it is sufficient to prove that the space is not complete.

**Example 35** Let \( X = (\mathbb{Q}, \| \cdot \|_1) \) i.e. set of rational numbers (\( \mathbb{Q} \)) with scalar field also as the set of rational numbers (\( \mathbb{Q} \)) and norm defined as

\[
\| x \|_1 = |x|
\]

A vector in this space is a rational number. In this space, we can construct Cauchy sequences which do not converge to a rational numbers (or rather they converge to irrational numbers). For example, the well known Cauchy sequence

\[
x^{(1)} = 1/1
\]

\[
x^{(2)} = 1/1 + 1/(2!)
\]

\[
\ldots \ldots
\]

\[
x^{(n)} = 1/1 + 1/(2!) + \ldots + 1/(n!)
\]

converges to \( e \), which is an irrational number. Similarly, consider sequence

\[
x^{(n+1)} = 4 - (1/x^{(n)})
\]

Starting from initial point \( x^{(0)} = 1 \), we can generate the sequence of rational numbers

\[3/1, 11/3, 41/11, \ldots \]
which converges to $2 + \sqrt{3}$ as $n \to \infty$. Thus, limits of the above sequences is outside the space $X$ and the space is incomplete.

**Example 36** Consider sequence of functions in the space of twice differentiable continuous functions $C^{(2)}(\mathbb{R})$:

$$f^{(k)}(t) = \frac{1}{2} + \frac{1}{\pi} \tan^{-1}(kt)$$

defined in interval $-\infty < t < \infty$, for all integers $k$. The range of the function is $(0,1)$. As $k \to \infty$, the sequence of continuous function converges to a discontinuous function

$$u^{(x)}(t) = \begin{cases} 0 & -\infty < t < 0 \\ 1 & 0 < t < \infty \end{cases}$$

**Example 37** Let $X = (\mathcal{C}(0,1), \| \cdot \|_1)$ i.e. space of continuous function on $[0,1]$ with one norm defined on it i.e.:

$$\| x(t) \|_1 = \int_0^1 |x(t)| dt \quad \text{------- (24)}$$

and let us define a sequence [2]

$$x^{(n)}(t) = \begin{cases} 0 & (0 \leq t \leq \left(\frac{1}{2}\right) - \frac{1}{n}) \\ n(t - \frac{1}{2}) + 1 & \left(\frac{1}{2}\right) - \frac{1}{n} \leq t \leq \frac{1}{2} \\ 1 & \left(t \geq \frac{1}{2}\right) \end{cases} \quad \text{------- (25)}$$

Each member is a continuous function and the sequence is Cauchy as

$$\| x^{(n)} - x^{(m)} \| = \frac{1}{2} \left| \frac{1}{n} - \frac{1}{m} \right| \to 0 \quad \text{------- (26)}$$

However, as can be observed from Figure 2, the sequence does not converge to a continuous function.

**Figure 2:** Sequence of continuous functions

The concepts of convergence, Cauchy sequences and completeness of space assume importance in the analysis of iterative numerical techniques. Any iterative numerical method generates a sequence of vectors and we have to assess whether the sequence is Cauchy to terminate the iterations. To a beginner, it may appear that the concept of incomplete vector space does not have much use in practice. It may be noted that, when we compute numerical solutions using any computer, we are working in finite dimensional incomplete vector spaces. In any computer with finite precision, any irrational number such as $\pi$ or $e$, is approximated by an rational number due to finite precision. In fact, even if we want to find a solution in $\mathbb{R}^n$, while using a finite precision computer to compute a solution,
we actually end up working in $Q^n$ and not in $R^n$. 