13.5 Steps to obtain the Root Locus

Note: Different books give these set of steps differently. We will follow something that is simple and works in most cases. You can take a look at D’Azzo and Houpis for a complete set of steps.

**Step 1:** Put the closed-loop system in Evan’s form.

\[ G_c(s) = \frac{G_1(s)}{1 + kG_2(s)} \]

where, \( k \) is the parameter to be varied. We assume that \( k \in [0, \infty) \).

**Step 2:** Find the starting \((k = 0)\) and the ending \((k = \infty)\) points.

(a) Locate the open loop poles and zeros. Let there be \( n \) open loop poles and \( m \) open loop zeros.

(b) Identify the number of loci. \((n \text{ poles} \Rightarrow n \text{ loci})\).

(c) Now the starting and ending points of the loci:

(i) As \( k \to 0 \), the loci approach the open loop poles.

\[ 1 + kG(s) = 0 \Rightarrow |kG(s)| = 1, \text{ for all } s \text{ on the root locus.} \]

\[ \Rightarrow G(s) \to \infty \text{ as } k \to 0. \]

\[ \Rightarrow \text{If } s_0 \text{ is on the root locus as } k \to 0 \text{ then } s_0 \text{ must be a pole of } G(s). \]

(ii) As \( k \to \infty \), \( m \) of the loci approach the open loop zeros, and the rest go to \( \infty \).

\[ 1 + kG(s) = 0 \Rightarrow |kG(s)| = 1, \text{ for all } s \text{ on the root locus.} \]

\[ \Rightarrow G(s) \to 0 \text{ as } k \to \infty. \]

Now, \( G(s) = \frac{N(s)}{D(s)} = \frac{s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{s^m + a_{m-1}s^{m-1} + \cdots + a_1 s + a_0} \)

and, \( m \leq n \).

So, for \( G(s) \to 0 \)

Either \( N(s) = 0 \Rightarrow s \text{ is a zero of } G(s). \)

Or, \( D(s) \to \infty \Rightarrow s \to \pm \infty. \)

**Step 3:** Find the real axis segments. That is, on the real axis, draw the root locus to the left of the odd number of open loop poles and zeros.

**Step 4:** Find the asymptotes.

The \( n - m \) poles that do not approach the zeros go to \( \infty \) along asymptotes.

(a) Find the asymptote angles.

\[ \phi_\alpha = \frac{180^\circ + \alpha \cdot 360^\circ}{n - m} = \frac{\pi + 2\alpha \pi}{n - m}, \quad \alpha = 0, 1, \ldots, n - m - 1 \]

\[ \text{Proof. As } s \to \infty, \]
\[ kG(s) = k \cdot \frac{s^m + \cdots + b_1 s + b_0}{s^n + \cdots + a_1 s + a_0} \cong k \cdot \frac{s^m}{s^n} = \frac{k}{s^{n-m}} \]

Now, \(1 + kG(s) = 0\) for all \(s\) on the root locus, so,
\[ kG(s) \cong \frac{k}{s^{n-m}} = -1 \]
\[ \Rightarrow s = (-k)^{\frac{1}{n-m}} \]

For \(k > 0\), we have
\[-k = ke^{j(\pi + \alpha 2\pi)} , \ \alpha = 0, 1, 2, \ldots\]

So,
\[ s = \left[ ke^{j(\pi + \alpha 2\pi)} \right]^{\frac{1}{n-m}} = k^{\frac{1}{n-m}} e^{j\left(\frac{\pi + \alpha 2\pi}{n-m}\right)} , \ \alpha = 0, 1, 2, \ldots, n - m - 1 \]

Why does \(\alpha\) go only up to \(n - m - 1\)? It is because the angles repeat after this point.

So, as \(k \to \infty\), \(n - m\) poles go to \(\infty\) along asymptotes with angles,
\[ \phi_\alpha = \frac{180^\circ + \alpha 360^\circ}{n - m} = \frac{\pi + 2\alpha\pi}{n - m} , \ \alpha = 0, 1, \ldots, n - m - 1 \]

An Example:

Let
\[ kG(s) =\frac{s + a}{(s^2 + 2\zeta \omega_n s + \omega_n^2)(s + b)^2} \]

Here, \(n = 4\), \(m = 1\).

As \(k \to \infty\), \(n - m = 3\) of the closed loop poles will go to \(\infty\). Only one of them will approach the zero \(s = -a\).

The asymptotes are given by,
\[ \phi_\alpha = \frac{180^\circ + \alpha 360^\circ}{n - m} , \ \alpha = 0, 1, 2 \]
\[ = \frac{180^\circ}{3}, \ \frac{180^\circ}{3}, \ \frac{180^\circ + 360^\circ}{3}, \ \frac{180^\circ + 720^\circ}{3} \]
\[ = 60^\circ, 180^\circ, 300^\circ \]
Figure 13.16: Open loop poles and zeros and the asymptotic centroid

Figure 13.17: Possible root loci

(b) Find the asymptotic centroid.

\[ \sigma = \frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{n - m} \]

where, \( p_i \) and \( z_i \) are the open loop poles and zeros.

**Proof.**

\[
kG(s) = k \cdot \frac{s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} = k \cdot \frac{(s - z_1)(s - z_2)\cdots(s - z_m)}{(s - p_1)(s - p_2)\cdots(s - p_n)}
\]

Note that,

\[ b_{m-1} = -\sum_{k=1}^{m} z_k \]
is the sum of the poles going to $\infty$.

But, note that $-(a_{n-1} - b_{m-1}) = \sum_{i=1}^{n-m} \beta_i$ is the sum of the poles going to $\infty$. But,

\[-a_{n-1} = \sum_{j=1}^{n} p_j\]

is the sum of the open loop poles, and

\[-b_{m-1} = \sum_{j=1}^{m} z_j\]

Continuing in this way, we can obtain,

\[kG(s) = k \cdot \frac{1}{s^{n-m} + (a_{n-1} - b_{m-1})s^{n-m-1} + \cdots} = -1\]

\[\Rightarrow s^{n-m} + (a_{n-1} - b_{m-1})s^{n-m-1} + \cdots + k = 0\]

Let the roots of the polynomial be given by,

\[s_i = \beta_i, \quad i = 1, 2, \ldots, n - m\]

Then,

\[(s - \beta_1)(s - \beta_2) \cdots (s - \beta_{n-m}) = s^{n-m} - \left(\sum_{i=1}^{n-m} \beta_i\right) s^{n-m-1} + \cdots + k\]

Then,

\[a_{n-1} - b_{m-1} = -\sum_{i=1}^{n-m} \beta_i\]

Try proving the above assertion yourself.

Now, let us do some long division (back to school!).

\[\frac{s^{n-m} + (a_{n-1} - b_{m-1})s^{n-m-1} + \cdots}{s^{n} + b_{m-1}s^{n-1} + \cdots + a_{n-1}s + a_0}\]

\[(a_{n-1} - b_{m-1})s^{n-1} + \cdots \]
is the sum of the open loop zeros.

Finally, the centroid is given by,

\[ \sigma = \frac{\sum_{i=1}^{n-m} \beta_i}{n - m} = \frac{- (a_{n-1} - b_{m-1})}{n - m} = \frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{n - m} \]

\[ = \frac{\sum \text{open loop poles} - \sum \text{open loop zeros}}{\text{Number of poles} - \text{Number of zeros}} \]

Note that the centroid has to be real since complex conjugate poles or zeros cancel the imaginary parts.

Also note that the whole proof depends on the fact that,

\[(s - f_1)(s - f_2) \cdots (s - f_n) = s^n + g_{n-1}s^{n-1} + \cdots + g_1s + g_0 \]

and

\[ g_{n-1} = - \sum_{i=1}^{n} f_i \]

An Example: Let the open loop plant transfer function be

\[ G(s) = \frac{1}{s(\tau s + 1)} \]

With P-control, the closed-loop transfer function would be,

\[ G_c(s) = \frac{k}{\tau s^2 + s + k} \]

which has poles,

\[ p_{1,2} = \frac{-1 \pm \sqrt{1 - 4\tau k}}{2\tau} \]

Asymptotic angles:

\[ m = 0, \ n = 2, \ n - m = 2. \]

So there are two asymptotes.

\[ \phi_\alpha = \phi_\alpha = \frac{180^\circ + \alpha 360^\circ}{n - m}, \ \alpha = 0, 1 \]

\[ = \frac{180^\circ}{2}, \ \frac{180^\circ + 360^\circ}{2} \]

\[ = 90^\circ, \ 270^\circ \]

Asymptotic centroid:
These are the same as that given in the figure.

Another Example: Let the open loop plant transfer function be,

\[ G(s) = \frac{1}{(s+4)^2 + 16} \]

With an integral control \( K(s) = \frac{k}{s} \), the closed-loop transfer function is,

\[ G_c(s) = \frac{k \cdot \frac{1}{(s+4)^2 + 16}}{1 + \frac{k}{s} \cdot \frac{1}{(s+4)^2 + 16}} \]

which is already in Evan’s form.

There are 3 open loop poles \((n = 3)\) and no open loop zeros \((m = 0)\).

\( p_1 = 0, \quad p_{2,3} = -4 \pm j4. \)

So, the asymptotic angles are,

\[ \phi_\alpha = \phi_\alpha = \frac{180^\circ + \alpha 360^\circ}{n-m}, \quad \alpha = 0, 1, 2 \]

\[ = \frac{180^0, 180^0 + 360^0, 180^0 + 720^0}{3}, \quad \frac{180^0 + 720^0}{3} = \frac{3}{3} \]

\[ = 60^\circ, 180^\circ, 300^\circ \]

The centroids are,

\[ \sigma = \sum \frac{p_i}{3} = \frac{0 + (-4 + j4) + (-4 - j4)}{3} = \frac{-8}{3} \]

The sketch of the root locus is shown in the figure below.
Step 5: Find the departure and arrival angles.

(a) The departure angle $\theta_{d_i}$ of the locus from pole $p_i$ is,

$$\sum_{k=1}^{m} \theta_{z_k} - \sum_{j=1, j \neq i}^{n} \theta_{p_j} - \theta_{d_i} = 180^\circ + l \cdot 360^\circ, \quad l = 0, \pm 1, \pm 2, \ldots$$

where $\theta_{z_k}$ is the angle made by a vector drawn from the zero $z_k$ to the pole $p_i$, and $\theta_{p_j}$ is the angle made by a vector drawn from pole $p_j$ to pole $p_i$. See the figure given below for illustration.

**Proof:** The departure angle expression is obtained directly from an application of the angle criterion. Take a point $s_i$ very close to $p_i$. 

![Figure 13.19: Root locus of the example problem](image1)

![Figure 13.20: Illustration for the departure angle](image2)
Let the angle made by the vector from the pole $p_i$ to the point $s_i$ be $\theta_{p_i}$. The angle made by the vectors from the other poles and zeros to $s_i$ is almost the same as the angles made by vectors from other poles and zeros to $p_i$ (since $s_i$ and $p_i$ are close). Now apply the angle criterion.

$$\sum_{k=1}^{m} \theta_{z_k} - \sum_{j=1, j \neq i}^{n} \theta_{p_j} - \theta_{p_i} = 180^\circ + l \cdot 360^\circ, \quad l = 0, \pm 1, \pm 2, \cdots$$

Now, solve for $\theta_{p_i} = \theta_{d_i}$.

**An Example:** Let

$$G(s) = \frac{1}{(s+4)^2 + 16}, \quad K(s) = \frac{k}{s}$$

Then,

$$K(s)G(s) = \frac{k}{s[(s+4)^2 + 16]}$$

Departure angle for pole $p_2$,

$$-(135^\circ + 90^\circ) - \theta_{d_2} = 180^\circ \Rightarrow \theta_{d_2} = -45^\circ \text{ or } 315^\circ$$
(b) The arrival angle $\theta_{a_i}$ of the locus at zero $z_i$ is computed from,

$$\sum_{k=1, k \neq i}^{m} \theta_{z_k} - \sum_{j=1}^{n} \theta_{p_j} + \theta_{a_i} = 180^\circ + l360^\circ, \ l = 0, \pm 1, \pm 2, \ldots$$

where, $\theta_{z_k}$ is the angle made by a vector drawn from zero $z_k$ to zero $z_i$ and $\theta_{p_j}$ is the angle made by a vector drawn from pole $p_j$ to zero $z_i$. See the figure below for illustration.

Figure 13.23: Illustration for the arrival angle

**Proof:** Similar to the previous proof.

**An Example:** Let

$$G(s) = \frac{(s - 4)^2 + 16}{(s + 4)^2 + 16}, \ K(s) = \frac{k}{s}$$

Then,

$$K(s)G(s) = \frac{k[(s - 4)^2 + 16]}{s[(s + 4)^2 + 16]}$$

Note that

$p_1 = 0, \ p_{2,3} = -4 \pm j4, \ z_{1,2} = 4 \pm j4.$

Figure 13.24: Arrival angle example
The angle $\theta_d$ is determined from,

$$(135^\circ + 180^\circ) - (135^\circ + 90^\circ) - \theta_d = 180^\circ \Rightarrow \theta_d = -90^\circ \text{ or } 270^\circ$$

The arriving angle $\theta_a$ is,

$$90^\circ - (45^\circ + 45^\circ) + \theta_a = 180^\circ \Rightarrow \theta_a = 180^\circ$$

**Step 6:** Find the real axis breakaway points.

Real axis locus segments that meet always break away at $\pm 90^\circ$. Generally, two approaching loci always meet at a relative angle of $180^\circ$, and then break away changing direction by $\pm 90^\circ$. Breakaway point locations can be difficult to solve for (in fact, this could be as difficult as solving for the exact root locus!).

As illustration, look at the example given for the asymptotic centroid computation.

**Step 7:** Use the Routh-Hurwitz criterion judiciously to identify the gain values for which poles cross from stable to unstable region. But remember that this may not always work, especially if there are poles already existing in the unstable region and you are trying to find out the value of $k$ for which some stable pole migrates to the unstable region. Some times the auxiliary equations and its roots help in this.