Chapter 11

Stability of Linear Systems

11.1 Some Definitions

The following definitions of stability are relevant to linear time invariant (LTI) systems.

Stable: An LTI system is stable if and only if its natural response approaches zero as time $\to \infty$.

Unstable: An LTI system is unstable if and only if its natural response grows without bound as time $\to \infty$.

Marginally stable: An LTI system is marginally stable if and only if its natural response neither grows nor approaches zero as time $\to \infty$ (for example, a sinusoidal response).

What is a natural response of a system?

The natural response of a system is the response which is not due to any input, but only that which is due to initial conditions.

An Example:

Consider a first-order system:

$$\dot{x} + ax = br$$

Taking Laplace transform on both sides,

$$sX(s) - x(0) + aX(s) = bR(s)$$

Let $r = 0$, then $R(s) = 0$.

$$\begin{aligned} X(s) &= \frac{x(0)}{s+a} \\ &\Rightarrow x(t) = x(0)e^{-at} \end{aligned}$$

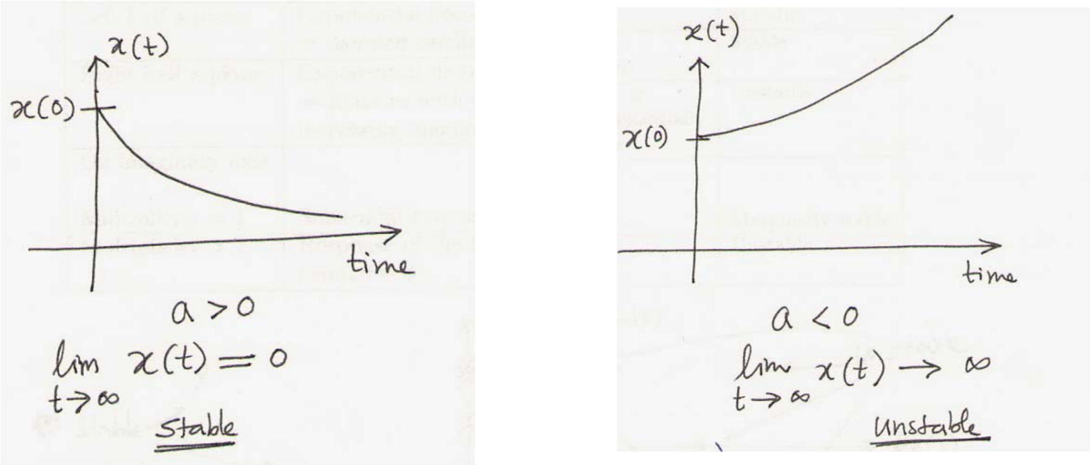

Figure 11.1: *Stable and unstable natural response of the example system*

This is the natural response of the system.

Another Example:

Consider the following system, when the input $r(t)$ is zero.

$$
\begin{aligned}
\ddot{x} \quad + \quad & \omega^2 x = br = 0 \\
& \Rightarrow s^2 X(s) - \dot{x}(0) - s x(0) + \omega^2 X(s) = 0 \\
& \Rightarrow X(s) = \frac{s x(0)}{s^2 + \omega^2} + \frac{\dot{x}(0)}{s^2 + \omega^2}
\end{aligned}
$$

So,

$$x(t) = x(0) \cos \omega t + \frac{\dot{x}(0)}{\omega} \sin \omega t$$

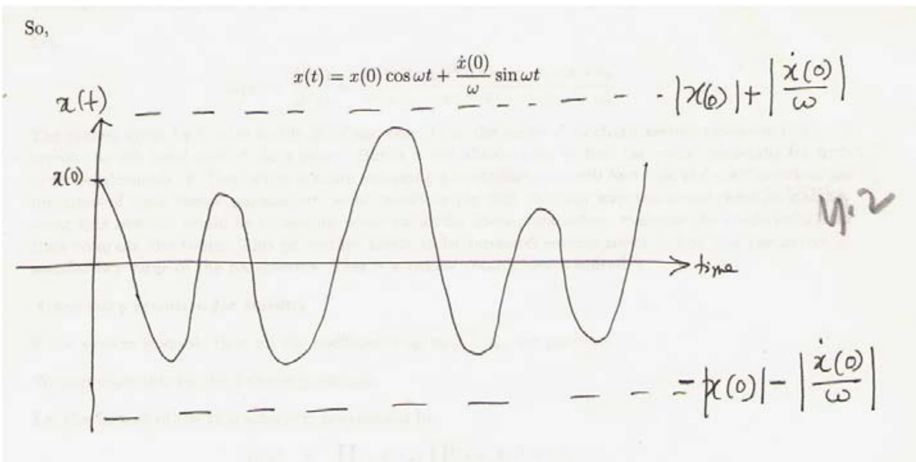

Figure 11.2: *Marginally stable natural response of the example system*

The system has bounded response, but is marginally stable.

S-plane interpretation of stability

Pole location	Natural Response	Stability
Left half s-plane	Exponential decay or damped oscillations	Stable
Right half s-plane	Exponential increase or oscillations with exponentially increasing magnitude	Unstable
On imaginary axis Multiplicity $= 1$ Multiplicity > 1	 Sinusoidal response Response of the type $t\sin(\omega t + \phi)$	 Marginally stable Unstable

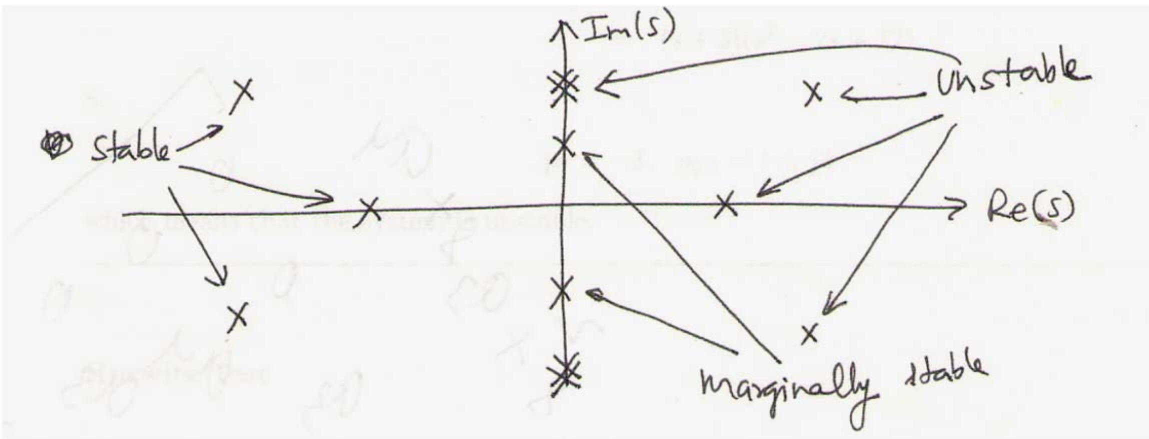

Figure 11.3: *Interpretation of stability in the s-plane*

11.2 Hurwitz Determinants

The Hurwitz stability test is the most useful for testing the stability of large-order LTI systems.

Let,

$$G(s) = \frac{N(s)}{D(s)} = \frac{s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

The system given by $G(s)$ is stable iff (if and only if) all the roots of its characteristic equation $D(s) = 0$ are on the left hand side of the s-plane. But it is not always easy to find the roots, especially for high order polynomials. In fact, when you are designing a controller, you will find that the coefficients a_i are functions of some design parameters, which would imply that the only way you could check for stability using this method would be to assume some values for these parameters, evaluate the coefficients, and then compute the roots. This procedure needs to be repeated several times before you can arrive at satisfactory value of the parameters. This is a cumbersome process indeed.

A necessary condition for stability.

If the system is stable then all the coefficients $a_0, a_1, \ldots, a_n$ are positive.

We can show this by the following analysis:

Let the factors of the characteristic polynomial be,

$$\begin{aligned} D(s) &= \prod_k (s+\alpha_k) \prod_j [(s+\beta_j)^2 + \gamma_j^2] \\ &= \prod_k (s+\alpha_k) \prod_j [(s^2 + 2\beta_j s + (\beta_j^2 + \gamma_j^2)] \end{aligned}$$

Since α_j and β_j are both positive (since the poles are on the left hand s-plane), the coefficients of $D(s)$ will all be positive.

Note that the above is not a sufficient condition. So,

Stability $\Rightarrow$ All coefficients of characteristic polynomial are positive.

All coefficients of characteristic polynomial are positive $\not\Rightarrow$ Stability.

Some coefficients of characteristic polynomial are negative or zero $\Rightarrow$ Unstable.

An Example:

$$\begin{aligned} D(s) &= s^3 + s^2 + 11s + 51 \\ &= (s+3)(s^2 - 2s + 17) \end{aligned}$$

So,

$$p_1 = -3, \quad p_{2,3} = 1 \pm j4$$

which means that the system is unstable.

Hurwitz Test

This test provides a necessary and sufficient condition for stability.

1. Let $G(s) = \frac{N(s)}{D(s)}$, where

$$D(s) = a_0 s^n + a_1 s^{n-1} + a_2 sn - 2 + \cdots + a_{n-1} s + a_n$$

(Note that the notation is slightly different).

2. Form the Hurwitz determinants as follows (put '0' for undefined coefficients):

$$
\begin{aligned}
D_1 &= a_1; \quad D_2 = \det\begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} \\
D_3 &= \det\begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix}; \quad D_4 = \det\begin{vmatrix} a_1 & a_3 & a_5 & a_7 \\ a_0 & a_2 & a_4 & a_6 \\ 0 & a_1 & a_3 & a_5 \\ 0 & a_0 & a_2 & a_4 \end{vmatrix} \\
&\vdots \\
D_n &= \det\begin{vmatrix} a_1 & a_3 & a_5 & a_7 & \cdots & 0 \\ a_0 & a_2 & a_4 & a_6 & \cdots & 0 \\ 0 & a_1 & a_3 & a_5 & \cdots & 0 \\ 0 & a_0 & a_2 & a_4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & a_n \end{vmatrix}
\end{aligned}
$$

   Note that an n-th order system will have n Hurwitz determinants.

3. The roots of $D(s)$ will all have negative real parts iff all the Hurwitz determinants $D_1, D_2, \ldots, D_n$ are positive. That is, none of them are zero or negative.

An Example:

$$
\begin{aligned}
D(s) &= s^3 + s^2 + 11s + 51 \\
&= a_0 s^3 + a_1 s^2 + a_2 s + a_3
\end{aligned}
$$

So, the Hurwitz determinants are,

$$
\begin{aligned}
D_1 &= a_1 = 1 \\
D_2 &= \det\begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} = \det\begin{vmatrix} 1 & 51 \\ 1 & 11 \end{vmatrix} = 11 - 51 = -40 < 0 \\
D_3 &= \det\begin{vmatrix} a_1 & a_3 & 0 \\ a_0 & a_2 & 0 \\ 0 & a_1 & a_3 \end{vmatrix} = \det\begin{vmatrix} 1 & 51 & 0 \\ 1 & 11 & 0 \\ 0 & 1 & 51 \end{vmatrix} = 1(11 \times 51) - 51(51) = -40 \times 51 < 0
\end{aligned}
$$

Since D_2 and D_3 are negative, the system is unstable. In fact,

$$s^3 + s^2 + 11s + 51 = (s+3)(s^2 - 2s + 17)$$

which has poles at,

$$p_1 = -3, \quad p_{2,3} = 1 \pm j4$$

Hurwitz determinants are not independent of each other. For example, for a 4th order system you don't need to compute D_4 to know its sign.

Consider a fourth order system,

$$D(s) = a_0 s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4$$

Then,

$$D_4 = \begin{vmatrix} a_1 & a_3 & 0 & 0 \\ a_0 & a_2 & a_4 & 0 \\ 0 & a_1 & a_3 & 0 \\ 0 & a_0 & a_2 & a_4 \end{vmatrix} = a_4 \begin{vmatrix} a_1 & a_3 & 0 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} = a_4 D_3$$

So, if $D_3 > 0$ and $a_4 > 0$ (as it should be when the system is stable) then $D_4 > 0$ too.

In fact, a general result says that you need to check only half the Hurwitz determinants (either the odd numbered ones or the even numbered ones).

Lienard-Chipart Criterion: A polynomial with real positive coefficients has roots with negative real parts if and only if either all the even Hurwitz detrminants are positive or all the odd Hurwitz determinants are positive.

11.3 Routh-Hurwitz Criterion

The Routh-Hurwitz criterion is an off-shoot of the Hurwitz determinants idea, but makes the computations much simpler through what is known as a Routh table.

The best way to demonstrate the formation of the Routh table is through an example. Consider a sixth order system.

$$D(s) = a_0 s^6 + a_1 s^5 + a_2 s^4 + a_3 s^3 + a_4 s^2 + a_5 s^1 + a_6$$

The Routh table or Routh array is formed as follows:

$$
\begin{array}{lllll}
s^6 & a_0 & a_2 & a_4 & a_6 \\
s^5 & a_1 & a_3 & a_5 & 0 \\
s^4 & \frac{a_1 a_2 - a_0 a_3}{a_1} = A & \frac{a_1 a_4 - a_0 a_5}{a_1} = B & \frac{a_1 a_6 - a_0 \times 0}{a_1} = a_6 & 0 \\
s^3 & \frac{A a_3 - a_1 B}{A} = C & \frac{A a_5 - a_1 a_6}{A} = D & \frac{A \times 0 - a_1 \times 0}{A} = 0 & 0 \\
s^2 & \frac{CB - AD}{C} = E & \frac{C a_6 - A \times 0}{C} = a_6 & 0 & 0 \\
s^1 & \frac{ED - C a_6}{E} = F & \frac{E \times 0 - C \times 0}{E} = 0 & 0 & 0 \\
s^0 & \frac{F a_6 - E \times 0}{F} = a_6 & 0 & 0 & 0
\end{array}
$$

The figure below shows the operation in the first two rows.

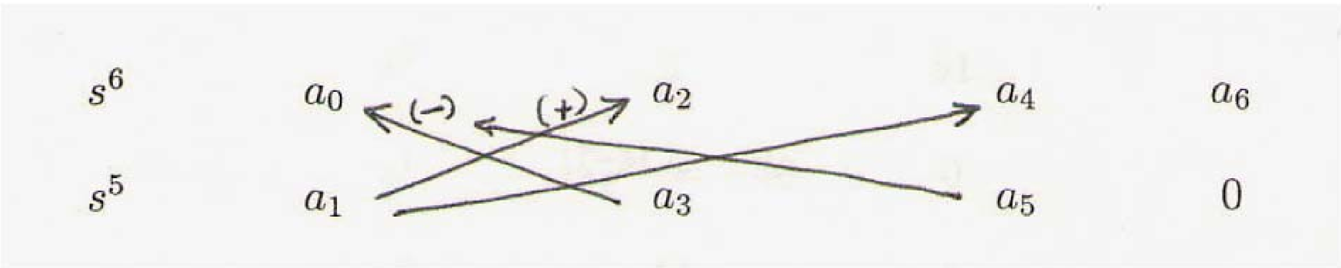

Figure 11.4: *Operation in the first two rows of the Routh table*

This is called the Routh array.

Now, look at the sign of the numbers in the first column of the Routh array.

1. If all the elements of the first column of the Routh array are of the same sign (usually positive) then all the roots are on the left hand side of the s-plane.
2. If there are changes in sign, then the number of sign changes indicates the number of roots with positive real parts.

What is the relation between the Hurwitz determinants and the Routh table?

It can be shown that for the sixth order system, each term in the first column of the Routh array has the following relationship with the corresponding Hurwitz determinants.

$$
\begin{aligned}
a_0 &= a_0 \\
a_1 &= D_1
\end{aligned}
$$

$$
\begin{aligned}
A &= \frac{D_2}{D_1} \\
C &= \frac{D_3}{D_2} \\
E &= \frac{D_4}{D_3} \\
F &= \frac{D_5}{D_4} \\
a_6 &= \frac{D_6}{D_5}
\end{aligned}
$$

An Example:

$$
\begin{aligned}
D(s) &= s^3 + s^2 + 11s + 51 \\
&= a_0 s^3 + a_1 s^2 + a_2 s + a_3
\end{aligned}
$$

The Routh array is,

$$
\begin{array}{lcc}
s^3 & 1 & 11 \\
s^2 & 1 & 51 \\
s^1 & \frac{11-51}{1} = -40 & 0 \\
s^0 & 51 & 0
\end{array}
$$

So, the system is unstable, and since there are two sign changes, there are two poles on the right hand side of the s-plane.

Remember that this system has poles

$$p_1 = -3, \quad p_{2,3} = 1 \pm j4$$