Lecture – 34

Stability Analysis of Nonlinear Systems Using Lyapunov Theory – II

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Outline

- Construction of Lyapunov Functions
- Definitions
  - Invariant Sets
  - Limit Sets
- LaSalle’s Theorem
Construction of Lyapunov Functions

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Variable Gradient Method:

* Select a $\nabla V = \frac{\partial V}{\partial X} = g(X)$ that contains some adjustable parameters

* Then $dV(X) = \left(\frac{\partial V}{\partial X}\right)^T dX$

$$\int_{\tilde{X}=0}^{X} dV(\tilde{X}) = \int_{\tilde{X}=0}^{X} \left(\frac{\partial V}{\partial \tilde{X}}\right)^T d\tilde{X}$$

$$V(X) - V(0) = \int_{\tilde{X}=0}^{X} g(\tilde{X}) d\tilde{X}$$

Note:
To recover a unique $V$, $\nabla V = g(X)$ must satisfy the "Curl Condition":

$$\frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i}$$

However, note that the intergal value depends on the initial and final states (not on the path followed). Hence, integration can be conveniently done along each of the co-ordinate axes in turn; i.e.
Variable Gradient Method:

\[ V(X) = \int_{0}^{x_1} g_1(\tilde{x}_1, 0, \ldots, 0) d\tilde{x}_1 \]
\[ + \int_{0}^{x_2} g_2(x_1, \tilde{x}_2, 0, \ldots, 0) d\tilde{x}_2 \]
\[ + \ldots \]
\[ + \int_{0}^{x_n} g_n(x_1, \ldots, x_{n-1}, \tilde{x}_n) d\tilde{x}_n \]

**Note:** The free parameter of \( g(X) \) are constrained to satisfy the symmetric condition, which is satisfied by all gradients of a scalar functions.
Theorem: A function $g(X)$ is the gradient of a scalar function $V(X)$ if and only if the matrix

$$\frac{\partial g(X)}{\partial X}$$

is symmetric; where

$$\begin{bmatrix}
\frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial g_n}{\partial x_1} & \cdots & \frac{\partial g_n}{\partial x_n}
\end{bmatrix}$$

Proof: Please see Marquez book (Appendix)
Variable Gradient Method:

**Proof**: (Necessity)

Assume: \( g(X) = \frac{\partial V}{\partial X} \)

\[
\frac{\partial g(X)}{\partial X} = \frac{\partial^2 V}{\partial X^2}
\]

\[
= \begin{bmatrix}
\frac{\partial^2 V}{\partial x_1^2} & \frac{\partial^2 V}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 V}{\partial x_1 \partial x_n} \\
\vdots & \ddots & \cdots & \vdots \\
\frac{\partial^2 V}{\partial x_n \partial x_1} & \frac{\partial^2 V}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 V}{\partial x_n^2}
\end{bmatrix}
\]
Variable Gradient Method:

\[ \frac{\partial^2 V}{\partial x_j \partial x_i} = \frac{\partial^2 V}{\partial x_i \partial x_j} \Rightarrow \frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i} \]

Hence, the matrix \[ \left[ \frac{\partial g(X)}{\partial X} \right] \] should be symmetric.
Variable Gradient Method:

Sufficiency: Assume \( \frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i} \)

\[
\begin{bmatrix}
\frac{\partial V}{\partial x_i} = g_i(X) & \forall i
\end{bmatrix}
\]
Variable Gradient Method:

We have:

\[ V(X) = \int_0^{x_1} g_1(\tilde{x}_1, 0, \ldots, 0) d\tilde{x}_1 \]

\[ + \int_0^{x_2} g_2(x_1, \tilde{x}_2, 0, \ldots, 0) d\tilde{x}_2 \]

\[ + \int_0^{x_n} g_n(x_1, x_2, \ldots, x_{n-1}, \tilde{x}_n) d\tilde{x}_n \]
Variable Gradient Method:

\[
\frac{\partial V}{\partial x_1} = g_1(x_1, 0, \ldots, 0) \\
+ \int_0^{x_2} \frac{\partial g_2}{\partial x_1}(x_1, \tilde{x}_2, 0, \ldots, 0) d\tilde{x}_2 \\
+ \int_0^{x_n} \frac{\partial g_n}{\partial x_1}(x_1, x_2, \ldots, x_{n-1}, \tilde{x}_n) d\tilde{x}_n \\
= g_1(x_1, 0, \ldots, 0) + \int_0^{x_2} \frac{\partial g_1}{\partial x_2}(x_1, \tilde{x}_2, 0, \ldots, 0) d\tilde{x}_2 + \\
+ \int_0^{x_n} \frac{\partial g_1}{\partial x_n}(x_1, x_2, \ldots, x_{n-1}, \tilde{x}_n) d\tilde{x}_n
\]
Variable Gradient Method:

\[
\begin{align*}
= & \quad g_1(x_1, 0, \ldots, 0) + g_1(x_1, \tilde{x}_2, 0, \ldots, 0)\bigg|_{\tilde{x}_2 = 0} \\
+ & \quad \cdots + g_1(x_1, x_2, \ldots, x_{n-1}, \tilde{x}_n)\bigg|_{\tilde{x}_n = 0} \\
= & \quad g_1(x_1, 0, \ldots, 0) + \left[ g_1(x_1, \tilde{x}_2, 0, \ldots, 0) - g_1(x_1, 0, \ldots, 0) \right] \\
+ & \quad \cdots + \left[ g_1(x_1, x_2, \ldots, x_n) - g_n(x_1, x_2, \ldots, x_n, 0) \right] \\
= & \quad g_1(x_1, x_2, \ldots, x_n) \\
\text{i.e} & \quad \frac{\partial V}{\partial x_1} = g_1(X) \\
\text{Similarly} & \quad \frac{\partial V}{\partial x_i} = g_i(X), \quad \forall i = 1, \ldots, n
\end{align*}
\]
Variable Gradient Method: Example

Problem: Analyze the stability behaviour of the following system

\[ \dot{x}_1 = -ax_1 \]
\[ \dot{x}_2 = bx_2 + x_1x_2^2 \]

Solution: \( X = 0 \) is an equilibrium point

Assume \( \frac{\partial V}{\partial X} = g(X) = \begin{pmatrix} k_1 & k \\ k & k_2 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \)

A symmetric matrix

\( \left( \text{Note: } \frac{\partial g_1}{\partial x_2} = \frac{\partial g_2}{\partial x_1} = k \right) \)
Variable Gradient Method: Example

Further, let us assume

\[ \frac{\partial V}{\partial X} = \begin{bmatrix} g_1(X) \\ g_2(X) \end{bmatrix} = \begin{bmatrix} k_1 x_1 \\ k_2 x_2 \end{bmatrix} \]

\[ \Rightarrow V(X) = \int_{0}^{x_1} g_1(\tilde{x}_1, 0) \, d\tilde{x}_1 + \int_{0}^{x_2} g_2(x_1, \tilde{x}_2) \, d\tilde{x}_2 \]

\[ = \int_{0}^{x_1} k_1 \tilde{x}_1 \, d\tilde{x}_1 + \int_{0}^{x_2} k_2 \tilde{x}_2 \, d\tilde{x}_2 \]

\[ = \frac{1}{2} \left( k_1 x_1^2 + k_2 x_2^2 \right) \]
Variable Gradient Method:

Choose \[ k_1, k_2 > 0 \]

Then \[ V(X) > 0 \quad \forall X \neq 0 \quad \text{and} \quad V(0) = 0 \]

\[ V(X) \] is a Lyapunov function candidate.

\[
\dot{V}(X) = g^T(X)f(X) = [k_1x_1 \quad k_2x_2] \begin{bmatrix} -ax_1 \\ b_2x_2 + x_1x_2^2 \end{bmatrix}
\]

\[
= -k_1ax_1^2 + k_2(b + x_1x_2)x_2^2
\]

Let us choose \[ k_1 = k_2 = 1. \] Then

\[
\dot{V}(X) = -ax_1^2 + (b + x_1x_2)x_2^2
\]
Variable Gradient Method:

Unless we know about $a$, $b$ at this point nothing can be said about $\dot{V}(X)$. Let us assume $a > 0$, $b < 0$. Then

$$\dot{V}(X) = -ax_1^2 - \left( |b| - x_1x_2 \right) x_2^2$$

$> 0$ (for small $x_1x_2$)

$\therefore \dot{V}(X) < 0$ in some domain $D \subset \mathbb{R}^2$ and $0 \in D$

i.e $\dot{V}(X)$ is negative definite in $D$

$\therefore$ The system is \underline{locally asymptotically stable}!
Krasovskii’s Method

Let us consider the system \( \dot{X} = f(X) \)

Let \( A(X) \triangleq \begin{bmatrix} \frac{\partial f}{\partial X} \end{bmatrix} \) : Jacobian matrix

**Theorem:**

If the matrix \( F(X) \triangleq A(X) + A^T(X) \) is **ndf** for all \( X \in D \) \((0 \in D)\), then the equilibrium point is **locally asymptotically stable** and a Lyapunov function for the system is

\[
V(X) = f^T(X)f(X)
\]

**Note:** If \( D = \mathbb{R}^n \) and \( V(X) \) is radially unbounded, then the equilibrium point is **globally asymptotically stable**.
Krasovskii’s Method

Claim-1: Since $F(X)$ is ndf, $A(X)$ is invertible.

Proof (by contradiction):

Let $A(X)$ be singular.

Then $\exists Y_0 \neq 0 : A(X)Y_0 = 0$

But $Y_0^T F Y_0 = Y_0^T (A + A^T) Y_0$

$= Y_0^T (AY_0) + (Y_0^T A^T) Y_0 = 0$

i.e. $F$ is not ndf.

Hence, $A(X)$ is non-singular (i.e., it is invertible).
Krasovskii’s Method

Claim-2: The invertibility (and continuity) of $A(X)$ guarantees that the function $f(X)$ can be uniquely inverted.

Justification:
This is perhaps straightforward from uniform convergence property of Taylor series expansion.

This leads to the conclusion that the dynamic system has only one equilibrium point in $D$. i.e. $f(X) \neq 0, \forall X \neq 0, X \in D$.

∴ $V(X) = f^T(X)f(X)$ is pdf.
Krasovskii’s Method

\[ \dot{V}(X) = f^T \dot{f} + \dot{f}^T f \]

\[ = f^T \left[ \frac{\partial f}{\partial X} \right]^T \dot{X} + \dot{X}^T \left[ \frac{\partial f}{\partial X} \right] f \]

\[ = f^T \left( A^T + A \right) f \]

\[ = f^T F f \]

Hence, if \( F(X) \) is negative definite, \( \dot{V}(X) \) is ndf.

So, by Lyapunov's theorem, \( X = 0 \) is asymptotically stable.
Krasovskii’s Method

**Note**: The global asymptotic stability of the system is guaranteed by the Global version of Lyapunov's direct method.

**Comment**: While the usage of this result is fairly straightforward, its applicability is limited in practice since $F(X)$ for many systems do not satisfy the negative definite property.
Generalized Krasovskii’s Theorem

**Theorem:**

Let \( A(X) \triangleq \begin{bmatrix} \frac{\partial f(X)}{\partial X} \end{bmatrix} \)

A sufficient condition for the origin to be asymptotically stable is that

\[ \exists \text{ two pdf matrices } P \text{ and } Q: \quad \forall X \neq 0, \text{ the matrix } \]

\[ F(X) = A^T P + PA + Q \]

is negative semi-definite in some neighbourhood \( D \) of the origin.

In addition, if \( D = \mathbb{R}^n \) and \( V(X) \triangleq f^T(X) P f(X) \) is radially unbounded, then the system is globally asymptotically stable.
Generalized Krasovskii’s Theorem

Proof: \[ V(X) = f^T(X)Pf(X) \]
\[ \dot{V}(X) = \left[ f^T P \dot{f} + \dot{f}^T P f \right] \]
\[ = f^T P \left( \frac{\partial f}{\partial X} \right)^T \dot{X} + \left( \frac{\partial f}{\partial X} \right)^T \dot{X} \right]^T Pf \]
\[ = f^T P A^T f + f^T A P f \]
\[ = f^T \left( P A^T + AP + Q - Q \right) f \]
\[ = f^T \left( P A^T + AP + Q \right) f - f^T Qf \]
\[ \leq 0 \text{ (ndf)} \quad \text{Hence, the result.} \]
Example

Problem: Analyze the stability behaviour of the following system

\[
\begin{align*}
\dot{x}_1 &= -6x_1 + 2x_2 \\
\dot{x}_2 &= 2x_1 - 6x_2 - 2x_2^3
\end{align*}
\]

Solution:

\[
A = \left[ \frac{\partial f}{\partial X} \right] = \begin{bmatrix}
-6 & 2 \\
2 & -6 - 6x_2^2
\end{bmatrix}
\]

\[
F = A + A^T = \begin{bmatrix}
-12 & 4 \\
4 & -12 - 12x_2^2
\end{bmatrix}
\]
Example

Eigenvalues of $F$:

\[
\begin{vmatrix}
\lambda + 12 & -4 \\
-4 & \lambda + 12 + 12x_2^2
\end{vmatrix} = 0
\]

\[
(\lambda + 12)^2 + (\lambda + 12)12x_2^2 - 16 = 0
\]

\[
\lambda^2 + 24\lambda + 144 + 12x_2^2\lambda + 144x_2^2 - 16 = 0
\]

\[
\lambda^2 + (24 + 12x_2^2)\lambda + (128 + 144x_2^2) = 0
\]

\[
\lambda_{1,2} = \frac{1}{2} \left[-(24 + 12x_2^2) \pm \sqrt{(24 + 12x_2^2)^2 - 4(128 + 144x_2^2)}\right]
\]
Example

\[ = - (12 + 6x_2^2) \pm \sqrt{(12 + 6x_2^2)^2 - (128 + 144x_2^2)} \]
\[ 0 < (*) < (12 + 6x_2^2) \]
\[ < 0 \quad \forall x_2 \in \mathbb{R} \]
\[ \therefore \ A \text{ is ndf in } \mathbb{R}^2 \]

Moreover, \[ V(X) = f^T(X) f(X) \]
\[ = (-6x_1 + 2x_2)^2 + (2x_1 - 6x_2 - 2x_2^3)^2 \]
\[ \rightarrow \infty \text{ as } \|X\| \rightarrow \infty \]
\[ \therefore \ X = 0 \text{ is globally asymptotically stable.} \]
Invariant Sets & La Salle’s Theorem

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Invariant Set

A set $M$ is said to be an "invariant set" with respect to the system $\dot{X} = f(X)$ if:

$X(0) \in M \implies X(t) \in M \ , \forall t > 0$

Examples:

(i) An equilibrium point ($M = X_e$)

(ii) Any trajectory of an autonomous system ($M = X(t)$)
Invariant Set

(iii) A limit cycle

(iv) \( M = \mathbb{R}^n \)

\[
\left\{ X \in \mathbb{R}^n : V(X) \leq l \right\}
\]

(v) \( \Omega_l = \left\{ \text{where, } V(X) \text{ is a continuously differentiable function} \right\} \)

\[
\left\{ \text{such that } \dot{V}(X) \leq 0 \text{ along the solution of } \dot{X} = f(X) \right\}
\]

Note: (1) \( V(X) \) need not be pdf.

(2) The condition implies that once the trajectory crosses the surface \( V(X) = c \), it can never come out again.
Limit Set

Definition:
Let \( X(t) \) be a trajectory of the dynamical system \( \dot{X} = f(X) \). Then the set \( N \) is called the limit set (or positive limit set) of \( X(t) \) if for any \( p \in N \), \( \exists \) a sequence of times \( \{t_n\} \in [0, \infty] \) such that \( X(t_n) \to p \) as \( t_n \to \infty \).

Note: Roughly, the limit set \( N \) of \( X(t) \) is whatever \( X(t) \) tends to in the limit.
Limit Set

Example:

(i) An asymptotically stable equilibrium point is the limit set of any solution starting from a close neighbourhood of the equilibrium point.

(ii) A stable limit cycle is the limit set for any solution starting sufficiently close to it.
Some Useful Results

Lemma-1:

If the solution \( X(t, t_0, X_0) \) of the system \( \dot{X} = f(X) \)
is bounded for \( t > t_0 \), then its limit set \( N \) is:

(i) bounded

(ii) closed \( \Rightarrow \) \( \text{(i.e. it is a non empty "compact set")} \)

(iii) Non-empty

Moreover, as \( t \to \infty \), the solution approaches \( N \).

Lemma-2: The limit set \( N \) of a solution \( X(t, t_0, X_0) \) of the autonomous
system \( \dot{X} = f(X) \) is invariant with respect to the same system.
A Useful Theorem (Subset of LaSalle’s Theorem)

Theorem: The equilibrium point $X = 0$ of the autonomous system $\dot{X} = f(X)$ is asymptotically stable if:

(i) $V(X) > 0$ (pdf) $\forall X \in D \quad [0 \in D]$

(ii) $\dot{V}(X) \leq 0$ (nsdf) in a bounded region $R \subset D$

(iii) $\dot{V}(X)$ does not vanish along any trajectory in $R$

other than the null solution $X = 0$

Moreover,

If the above conditions hold good for $R = \mathbb{R}^n$ and $V(X)$ is radially unbounded,

then $X = 0$ is globally asymptotically stable.
Proof of the Theorem

\[ \dot{V} \leq 0 \]

\[ \implies \text{The system is stable} \]

i.e. for each \( \varepsilon > 0 \), \( \exists \delta > 0 : \)

\[ \|X_0\| < \delta \implies \|X(t)\| < \varepsilon \]

or, Any solution starting inside the closed ball \( B_\delta \)

will remain within the closed ball \( B_\varepsilon \)

\[ \implies \text{The solution (starting within } B_\delta) \text{ is bounded.} \]
Proof of the Theorem

Hence, \( X(t) \) tends to its limit set \( N \subset B_\varepsilon \)
and \( B_\varepsilon \) is compact. (By Lemma - 1)
Moreover, \( V(X) \) is continuous on the compact set \( B_\varepsilon \)
and \( \dot{V}(X) \leq 0 \), \( \therefore V(X) \to L \geq 0 \) as \( t \to \infty \)
i.e \( V(X) = L \quad \forall X \in N \) (\( N \): the limit set)
Note that \( N \) is invariant set with respect to the
system \( \dot{X} = f(X) \) (By Lemma - 2)
Proof of the Theorem

⇒ Any solution that starts in $N$ will remain within it for all future time.

However, along that solution $\dot{V}(X) = 0$, as $V(X) = L$

But, by the assumption of the theorem, $\dot{V}(X)$ does not vanish along any trajectory other than the null solution $X = 0$

Hence, Any solution starting in $R \subset B_\delta$ converges to $X = 0$ as $t \to \infty$
Example – 1: Pendulum with Friction

Example: (Pendulum with friction)

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{g}{l}\sin x_1 - \left(\frac{k}{m}\right)x_2
\end{align*}
\]

\[
V(X) = \frac{1}{2}ml^2x_2^2 + mgl(1 - \cos x_1) > 0 \quad \forall X \in D = (-\pi, \pi) \times \mathbb{R}
\]

\[
\dot{V}(X) = -kl^2x_2^2 : \text{nsdf} \quad \text{[Note: } 0 \in D]\n\]
Example – 1: Pendulum with Friction

Now let us examine the condition

\[ \dot{V}(X) = 0 \quad \forall t \]

\[ -kl^2x_2^2 = 0 \]

\[ \Leftrightarrow \quad x_2 = 0 \quad \forall t \quad \Rightarrow \quad \dot{x}_2 = 0. \quad \text{Hence} \]

\[ \frac{g}{l} \sin x_1 + \frac{k}{m} x_2 = 0 \]

\[ \sin x_1 = 0 \quad (\therefore x_2 = 0) \quad \Rightarrow \quad x_1 = 0 \quad [\text{as} \quad x_1 \in (-\pi, \pi)] \]

Hence, \( \dot{V}(X) \) happens only for \( X = 0 \).

Hence, \( X = 0 \) is locally asymptotically stable!
Example – 2

Example: \[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_2 - \alpha x_1 - (x_1 + x_2)^2 x_2
\end{align*}
\]

Solution: Let \( V(X) = \alpha x_1^2 + x_2^2 \), \( \alpha > 0 \)

\[
\dot{V}(X) = \left( \frac{\partial V}{\partial X} \right)^T f(X)
\]

\[
\dot{V}(X) = \begin{bmatrix} 2\alpha x_1 & 2x_2 \end{bmatrix} \begin{bmatrix} x_2 \\
-x_2 - \alpha x_1 - (x_1 + x_2)^2 x_2 \end{bmatrix}
\]

\[
= 2\alpha x_1 x_2 - 2x_2^2 - 2\alpha x_1 x_2 - 2(x_1 + x_2)^2 x_2^2
\]
Example – 2

\[ \dot{V}(X) = -2x_2^2 \left[ 1 + (x_1 + x_2)^2 \right] \leq 0 \quad \text{(nsdf)} \]

Now \[ \dot{V}(X) = 0 \quad \forall t \]

\[ \Leftrightarrow x_2(t) = 0 \quad \forall t \]

\[ \Rightarrow \dot{x}_2 = 0 \]

\[ -x_2 - \alpha x_1 - (x_1 + x_2)^2 x_2 = 0 \quad \text{(However, } x_2 = 0) \]

\[ \therefore x_1 = 0 \quad \text{i.e. } X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]
Example – 2

Here we have:

(i) $\dot{V}(X)$ does not vanish along any trajectory other than $X = 0$

(ii) $\dot{V} \leq 0$ in $\mathbb{R}^n$

(iii) $V(X)$ is radially unbounded,

Hence, the origin is Globally asymptotically stable.
LaSalle’s Theorem

Let $V : D \rightarrow \mathbb{R}$ be a continuously differentiable (not necessarily pdf) function and (i) $M \subset D$ be a compact set, which is invariant with respect to the solution of $\dot{X} = f(X)$

(ii) $\dot{V} \leq 0$ in $M$

(iii) $E = \{ X : X \in M \text{ and } \dot{V}(X) = 0 \}$

i.e. $E$ is the set of all points of $M : \dot{V} = 0$

(iv) $N$ is the largest invariant set in $E$

Then Every solution starting in $M$ approaches $N$ as $t \rightarrow \infty$.  

ADVANCED CONTROL SYSTEM DESIGN
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Lasalle’s Theorem

Remarks:

(i) $V(X)$ is required only to be continuously differentiable
It need not be positive definite.

(ii) LaSalle's Theorem applies not only to equilibrium
points, but also to more general dynamic
behaviours such as limit cycles.

(iii) The earlier theorems (on asymptotic stability) can
be derived as a corollary of this theorem.
References

Thanks for the Attention...!