Chapter 8

Dynamic stability analysis – II – Longitudinal motion

(Lectures 28 to 32)

Keywords : Stability quartic or characteristic equation for longitudinal motion and its solution ; roots of characteristic equation and types of motions indicated by them ; short period oscillation (SPO) and long period oscillation (LPO) or Phugoid; equations of motion in state space or state variable form ; approximations for SPO and LPO; stability diagrams – one parameter, two parameter and root locus plot ; eigen values and eigen vectors, longitudinal stick-free dynamic stability.

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Example 8.1

8.1 Introduction

The small perturbation equations for the longitudinal motion are derived in the previous chapter. These are reproduced below for ready reference.

$$(\frac{d}{dt}-X_u)\Delta u - X_w\ \Delta w + g\cos\theta_0\ \Delta\theta = X_{\delta e}\ \Delta\delta_e + X_{\delta T}\ \Delta\delta_T \qquad (7.85)$$

$$-Z_u\Delta u + [(1 - Z_{\dot{w}})\frac{d}{dt} - Z_w]\Delta w - [(u_0+Z_q)\frac{d}{dt} - g\sin\theta_0]\Delta\theta$$

$$= Z_{\delta e}\Delta\delta_e + Z_{\delta T}\ \Delta\delta_T \qquad (7.86)$$

$$-M_u\Delta u - (M_{\dot{w}}\frac{d}{dt}+M_w)\Delta w + (\frac{d^2}{dt^2}-M_q\frac{d}{dt})\Delta\theta$$

$$= M_{\delta e}\ \Delta\delta_e + M_{\delta T}\ \Delta\delta_T \qquad (7.87)$$

In this chapter the longitudinal dynamic stability is analysed by examining whether the disturbances Δu, Δw and $\Delta\theta$ grow or subside with time. The stick-fixed case is considered initially. In this case, the elevator deflection does not change during the motion which follows after the disturbance. Stick-free case is dealt with in section 8.15.

8.2 Examination of stability of longitudinal motion - obtaining characteristic equation

After deriving the linearized equation for small perturbation, it was mentioned at the end of subsection 7.9.3, that the stability of the motion can be examined without obtaining the solution of the governing differential equations. The procedure to examine the stability is as follows.

The small perturbation equations (Eqs.7.85 to 7.87) are linear i.e. they do not involve terms containing products of dependent variables or their powers. Such a set of equations admits a solution of the form:

$$\Delta u = \rho_1 e^{\lambda t},\ \Delta w = \rho_2 e^{\lambda t},\ \Delta\theta = \rho_3 e^{\lambda t}. \tag{8.1}$$

Substituting for Δu, Δw and $\Delta\theta$ from Eq. (8.1) in Eqs.(7.85),(7.86), (7.87) and noting that for stick-fixed stability problem $\Delta\delta_e$ and $\Delta\delta_t$ are zero, gives the following equations :

$$\lambda \rho_1 e^{\lambda t} - X_u \rho_1 e^{\lambda t} - X_w \rho_2 e^{\lambda t} + g\cos\theta_0 \rho_3 e^{\lambda t} = 0 \tag{8.2}$$

$$- Z_u \rho_1 e^{\lambda t} + [(1-Z_{\dot w})\lambda - Z_w] \rho_2 e^{\lambda t} - [(u_0+Z_q)\lambda - g\sin\theta_0] \rho_3 e^{\lambda t} = 0 \tag{8.3}$$

$$- M_u \rho_1 e^{\lambda t} + (M_{\dot w}\lambda + M_w)\rho_2 e^{\lambda t} + (\lambda^2 - M_q\lambda)\rho_3 e^{\lambda t} = 0 \tag{8.4}$$

Dividing by $e^{\lambda t}$ the above equations can be rewritten as:

$$(\lambda - X_u)\rho_1 - X_w \rho_2 + g\cos\theta_0 \rho_3 = 0 \tag{8.5}$$

$$- Z_u \rho_1 + \{(1-Z_{\dot w})\lambda - Z_w\}\rho_2 + \{(u_0 - Z_q)\lambda - g\sin\theta_0\}\rho_3 = 0 \tag{8.6}$$

$$- M_u \rho_1 + (M_{\dot w}\lambda + M_w)\rho_2 + (\lambda^2 - M_q\lambda)\rho_3 = 0 \tag{8.7}$$

The Eqs. (8.5),(8.6) and (8.7) are a set of homogeneous equations for ρ_1, ρ_2 and ρ_3 . The solution $\rho_1 = \rho_2 = \rho_3 = 0$ is called a trivial solution for obvious reasons. For a non-trivial solution to exist, the following condition must be satisfied:

$$\begin{vmatrix} \lambda - X_u & -X_w & g\cos\theta_0 \\ -Z_u & (1-Z_{\dot w})\lambda - Z_w & (u_0 - Z_q)\lambda - g\sin\theta_0 \\ -M_u & M_{\dot w}\lambda + M_w & \lambda^2 - M_q\lambda \end{vmatrix} = 0 \tag{8.8}$$

When the determinant in Eq.(8.8) is expanded, it yields the following fourth degree polynomial in λ which is called the characteristic equation of the dynamic system

$$A\lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E = 0 \tag{8.9}$$

Equation(8.9) is also called stability quartic.

When Z_q and $Z_{\dot w}$ are ignored and θ_0 is taken zero, the coefficients A,B,C,D and E in Eq.(8.9) are :

$$A = 1$$
$$B = -M_q - u_0 M_{\dot{w}} - Z_w - X_u$$
$$C = Z_w M_q - u_0 M_w - X_w Z_u + X_u(M_q + u_0 M_{\dot{w}} + Z_w) \qquad (8.10)$$
$$D = -X_u(Z_w M_q - u_0 M_{\dot{w}}) + Z_u(X_w M_q + g M_{\dot{w}}) - M_u(u_0 X_w - g)$$
$$E = g(Z_u M_w - Z_w M_u)$$

8.3 Responses indicated by roots of characteristic equation

Equation (8.10) has four roots namely λ_1,λ_2, λ_3 and λ_4 . The response to the disturbance i.e. the variations of Δu, Δw and Δθ with time can now be written as:

$$\Delta u = \rho_{11} e^{\lambda_1 t} + \rho_{12} e^{\lambda_2 t} + \rho_{13} e^{\lambda_3 t} + \rho_{14} e^{\lambda_4 t} \qquad (8.11)$$

$$\Delta w = \rho_{21} e^{\lambda_1 t} + \rho_{22} e^{\lambda_2 t} + \rho_{23} e^{\lambda_3 t} + \rho_{24} e^{\lambda_4 t} \qquad (8.12)$$

$$\Delta \theta = \rho_{31} e^{\lambda_1 t} + \rho_{32} e^{\lambda_2 t} + \rho_{33} e^{\lambda_3 t} + \rho_{34} e^{\lambda_4 t} \qquad (8.13)$$

To evaluate the coefficients ρ_{11}, ρ_{12},.. , ρ_{34} the differential Eqs.(7.85) to (7.87) need to be solved with appropriate initial conditions. However, to examine the stability, it is enough to know the values of λ_1 to λ_4. Because the term $e^{\lambda t}$, which depends on λ ultimately decides whether the disturbances Δu, Δw and Δθ die down, remain same or increase with time. This is explained below.

The roots (λ_1 to λ_4) can be of the following six types:

a) λ is real and positive = r

b) λ is real and negative = -r

c) λ is zero

When the roots are complex they appear as a pair of complex conjugates (r+is) and (r-is) . Where 'r' is the real part, 's' is the complex part and 'i' is (√-1) .The two roots together are represented as (r±is). There could be three cases when the roots are complex.

d) λ_1 and λ_2 constitute a complex pair r± is with 'r' positive.

e)λ_1 and λ_2 constitute a complex pair r± is with 'r' negative.

f) λ_1 and λ_2 constitute a complex pair r± is with 'r' being zero.

The variations of $e^{\lambda t}$ with time, for the above six cases are explained below and shown in Fig.8.1.

I) When the root is a real number and positive the term $e^{\lambda t}$ becomes e^{rt}. It is evident that the disturbance(e.g. Δu) will grow exponentially with time (Fig. 8.1a). This response is called divergence.

II) When the root is a real number and is negative the term $e^{\lambda t}$ becomes e^{-rt}. This indicates that the disturbance will die down eventually (Fig.8.1b). This motion is called subsidence.

III) If the root is zero the term $e^{\lambda t}$ would become e^0 or unity. This indicates that the system would remain in the disturbed position (Fig.8.1c). This response is called neutral stability.

IV) When the roots form a complex pair, they appear as (r ± i s). There are following two possibilities.

a) When the four roots consist of two real roots (λ_1 and λ_2) and a complex pair (r± is), then the response would be of the form:

$$\Delta u = \rho_{11}\, e^{\lambda_1 t} + \rho_{12}\, e^{\lambda_2 t} + \rho'_{13}\, e^{rt} \cos\,(s\,t + C_1) \qquad (8.14)$$

where, ρ'_{13} and C_1 are constants.

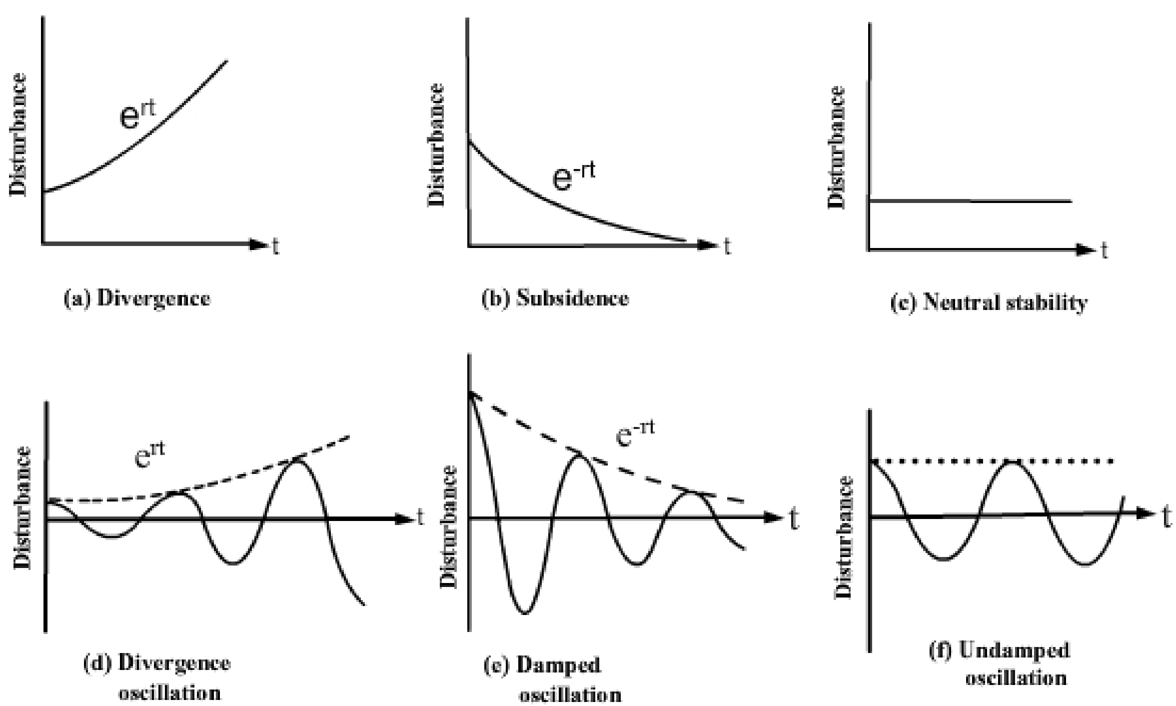

Fig.8.1 Motions following disturbance - as indicated by roots

Or $\Delta u = \rho_{11}\, e^{\lambda_1 t} + \rho_{12}\, e^{\lambda_2 t} + \rho_{13}\, e^{rt}\cos(s\,t) + \rho_{14}\, e^{rt}\sin(s\,t)$ (8.15)

It is observed that the response corresponding to the complex root is an oscillatory motion.

b) When the four roots consist of two complex pairs, ($r_1 \pm i\, s_1$) and ($r_2 \pm i\, s_2$), then the response is:

$$\Delta u = \rho'_{11} e^{r_1 t}\cos(s_1 t + C_1) + \rho'_{12} e^{r_2 t}\cos(s_2 t + C_2) \quad (8.16)$$

$$\text{Or } \Delta u = \rho_{11}\, e^{r_1 t}\cos(s_1 t) + \rho_{12}\, e^{r_1 t}\sin(s_1 t) + \rho_{13}\, e^{r_2 t}\cos(s_2 t) + \rho_{14}\, e^{r_2 t}\sin(s_2 t) \quad (8.17)$$

The amplitude of the oscillation is decided by the quantity ‘r’. Following three responses are possible depending on the value of ‘r’.

i) If ‘r’ is positive, then the amplitude of the periodic variation increases with each oscillation (Fig.8.1d). This mode is called divergent oscillation.

ii) If ‘r’ is negative, then the amplitude of the periodic variation decreases with each oscillation (Fig.8.1e). This mode is called damped oscillation.

iii) If ‘r’ is zero, then the amplitude of the periodic motion remains constant (Fig.8.1f). This mode is called undamped oscillation.

8.4 Types of roots which indicate dynamic stability

From the above discussion it is observed that for an equilibrium state to be dynamically stable, the roots of the characteristic equation have to be one of the following two types.

(a) When the root is real number, it should be negative.

(b)When the root is complex number, the real part should be negative.

Thus, the dynamic stability of the airplane can be judged by observing the roots of the characteristics equation. It is not necessary to obtain the response of the airplane to the disturbance. To illustrate the aforesaid discussion, Example 8.1 considers the dynamic stability of the general aviation airplane.

Example 8.1

Examine the dynamic stability of the general aviation airplane details given below. Figure 8.2 presents for the three-view drawing of the airplane.It may be pointed out that this example is adapted from Ref.2.4 and the airplane considered is same as in example 2.4 in chapter 2.It is called "Navion" in Ref 2.4.

Flight condition: Steady level flight at sea level at

u_0 = 176 ft/s = 53.64 m/s (M = 0.158)

Weight of the airplane = 12232.6 N, Mass of the airplane = m= 1247.4 kg.

Acceleration due to gravity (g) is taken as 9.80665 m / s^2, I_{yy} = 40675.8 kg m^2.

Geometric details:

S = 17.09 m^2, $\bar{c}$ = 1.737 m, b = 10.18 m

Other details:

ρ = 1.225 kg m^{-3} , C_L = 0.41, C_D = 0.05

$C_{L\alpha}$ = 4.44 rad^{-1}, $C_{D\alpha}$ = 0.33 rad^{-1}, $C_{m\alpha}$ = - 0.683 rad^{-1},

C_{Lu} = 0 , C_{Du} = 0 , C_{mu} = 0 , $C_{m\dot{\alpha}}$ = - 4.36, C_{Lq} = 3.8 (from Ref.1.1, chapter 4) ,

C_{mq} = - 9.96 .

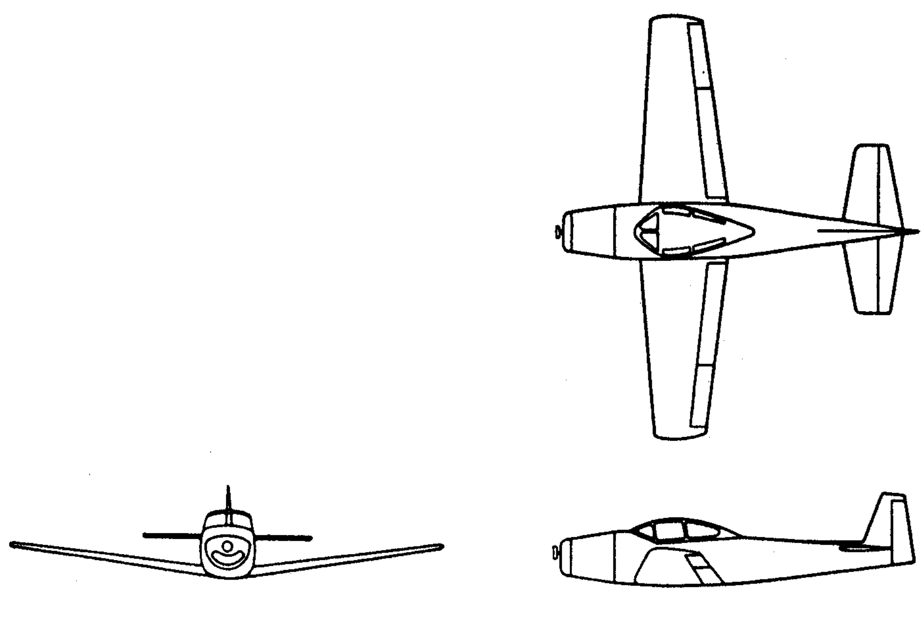

Fig.8.2 Three-view drawing of the general aviation airplane Navian
(Adapted from Ref.2.4)

Solution:

From the above data the following quantities, needed for obtaining stability derivatives, are deduced.

$$Q = \frac{1}{2}\rho u_0^2 = \frac{1}{2}\times 1.225\ (53.64)^2 = 1762.3\ Nm^{-2}, QS = 30117.7\,N,$$

$$Q S\bar{c} = 52314.4\ Nm,\ \frac{\bar{c}}{2u_0} = 0.0162\ s,\ \ mu_0 = 66910.5\ kg\ ms^{-1}$$

$$\frac{QS}{mu_0} = 0.450\ ,\ \frac{QS\bar{c}}{u_0 I_{yy}} = 0.240$$

The longitudinal stability derivatives are:

$$X_u = -\frac{QS}{mu_0}(C_{Du}+2C_D) = -0.45(0+2\times 0.05) = -0.045\ s^{-1}$$

$$Z_u = -\frac{QS}{mu_0}(C_{Lu}+2C_L) = -0.45(0+2\times 0.41) = -0.369\,s^{-1}$$

$$M_u = 0$$

$$X_w = -\frac{QS}{mu_0}(C_{D\alpha}-C_L) = -0.45(0.33-0.41) = 0.036\,s^{-1}$$

$$Z_w = -\frac{QS}{mu_0}(C_{L\alpha}+C_D) = -0.45(4.44+0.05) = -2.02\,s^{-1}$$

$$M_w = \frac{QS\bar{c}}{u_0 I_{yy}}C_{m\alpha} = 0.240\ (-0.683) = -0.164\,m^{-1}\,s^{-1}$$

$$X_{\dot{w}} = 0$$

$$Z_{\dot{w}} = 0$$

$$M_{\dot{w}} = -C_{m\dot{\alpha}}\frac{\bar{c}}{2u_0}\frac{QS\bar{c}}{u_0 I_{yy}} = -4.36\times 0.0162\times 0.240 = -0.01695\ m^{-1}$$

$$X_q = 0$$

$$Z_q = 0$$

$$M_q = C_{mq}\frac{\bar{c}}{2u_0}\frac{QS}{I_{yy}} = -9.96\times 0.0162\times 53.64\times 0.240 = -2.077\,s^{-1}$$

Substituting the numerical values of the stability derivatives in Eqs.(8.10) gives:

A = 1, B = 5.05, C = 13.15, D = 0.6735 and E = 0.593.

Hence, the stability quartic (Eq.8.9) or the characteristic equation appears as:

$$\lambda^4 + 5.05\lambda^3 + 13.15\lambda^2 + 0.6735\lambda + 0.593 = 0 \quad (8.18)$$

The roots of this equation are obtained in the next section.