In the previous lecture we were discussing about angular deformation in the flow. Now we will discuss about the Linear and Volumetric Deformation. So, because we are thinking of small time and small deformation, then actually if the deformation is integrated over time it will be a large deformation and that is what is the characteristic of the fluid. But in small time you have small deformation and in that respect you can decouple the linear and the angular deformation for your analysis.

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So, with that understanding, let us assume that you have a rectangular volume like this with the dimensions delta x, delta y and delta z. So, now, we will try to individually calculate what is the change in delta x, what is the change in delta y, what is the change in delta z and accordingly what is the change in volume of the fluid element.

So, to calculate the change in delta x let us just consider one of these edges. We separately draw the figure like this let us say this is A B this is delta x. So, after a time delta t A goes to A prime and B goes to say B prime. So, what is this? So, let us say u is the x component of velocity. So, this is u into delta t and this is u plus.
So, the net change in length of \( \Delta x \) is \( \Delta x \) plus the difference. So, the new \( \Delta x \) is old \( \Delta x \) plus this difference. So, that is ok. So, what is the rate of strain along \( x \)? I will later on tell you why these two indices are needed and all those things, but for the time being let us assume that I mean once we put it in symbols, we will understand that these are not really simple scalar quantities neither they are vectors, these are second order tensors.

So, what are second order tensors we will discuss in more details when we talk about the stressed tensor, but in this context I would like to mention that just roughly you know just to give you a rough picture. A second order tensor in the Cartesian notation will require two indices for its specification. So, that is why the two indices. Why are these two indices necessary? One index to describe the direction of the stress or strain whatever and another index to specify the normal direction of the plane which is used as a reference to calculate the stress or strain.

So, for example; so, if you consider want to calculate stress at a point. So, you calculate force per unit area. So, to calculate the force you imagine an area. So, there is a normal direction to the area, but there is also a direction of the force. So, the force is a vector, but that vector itself depends on the direction of the area which is considered. So, it is an augmented form of a vector. So, that requires two indices for its specification and it is called as a second order tensor.

So, formally in terms of mathematics a second order tensor, maps a vector on to a vector and we will see later on that how this is satisfied for the stressed tensor (Refer Time: 05:58) is the change in length, per unit length and this is the rate of deformation; so, per unit time. Limit as \( \Delta t \) tends to 0 and once you do that the higher order term will be gone. So, this will become.

Similarly so, these are the linear rates of strain. For fluid it is not really the change of length that people bother, it is the change in volume because say you have a container, you put the fluid in the container what do you fundamentally try to measure is what is the volume not that what are individual lengths. Lengths are very abstract quantities for fluid, but volume is the real practical quantity with which you deal.
So, let us translate this into volume. So, what is the new volume? New volume is new delta x which is this one into new delta y into new delta z. So, new delta x is delta x into 1 plus, this is a new delta x multiplied by new delta y, multiplied by new delta z.

So, then we can calculate these very systematically, we first multiply these two terms. So, if you do that, let me write it here delta x delta y 1 into 1 1 plus 1 into this plus this into 1 plus the product of these two terms, which is a smaller term which in technically is called as a higher order term in the series because it is delta t into delta t delta t square of the order of delta t square. Delta t being small, delta t square is one order smaller than delta t and other higher order terms are also neglected. So, we do not write that simply write dot dot dot.

Then we multiply that with this one delta z, delta z term. So, this becomes delta x delta y delta z times now this entire thing multiplied by 1. So, 1. So, this del w del z delta t how does it come? It is multiplied with this 1, other terms of the order of delta t square and they are neglected. So, all are higher order terms.

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So, what is the rate of volumetric strain? Change in volume, rate of volumetric strain. This is change in volume by original volume per unit time. So, the change in volume is this minus delta x into delta y into delta z. So, this therefore is there. When you calculate per unit time that is divide that by time, all other higher order terms will be 0. So, you will be left with.
So, this is nothing but the divergence of the velocity vector. So, a flow is said to be incompressible when there is no change in volume of a fluid element and then divergence of the velocity vector is equal to 0. So, one important question or one important point that needs to be remembered here. For a long time or in fact, it has become a tradition that this equation is posed as a consequence of conservation of mass or continuity for incompressible flow.

But now we can see that without referring to conservation of mass purely from kinematic constraints of volume we can derive this expression. So, for incompressible flow divergence of velocity equal to 0 does not necessarily follow from conservation of mass, but can follow just from pure kinematic constraints. It is a very special concept that needs to be clarified to the students typically in a little bit advanced level. Because otherwise students develop an impression that this is a consequence of continuity equation which we will discuss in a moment.

Now we have discussed all sorts of deformation; linear, volumetric angular, angular also shear and rotation everything we have discussed. Can you tell what is the common thing in all these expressions? It is essentially a spatial partial derivative of some component of velocity, right? So, the rate of deformation the generalized rate of deformation can be expressed in this form.

This is an index notation where i is equal to 1 means x, i is equal to 2 means y and i is equal to 3 means z. So, u 1 means u, u 2 means v and u 3 means w. This is just an alternative way called as index notation to you know just in a very compact way described all the derivatives; partial derivatives. So, this is called as the generalized rate of deformation tensor. This is also a second order tensor it requires two components for its two indices for its specification.

So, this you can write; so, we can decompose these it into two parts. So, what is this? This is deformation, why simply deformation? Because it could be linear or angular depending on whether i is equal to j or i not equal to j. So, that is why I am just writing deformation, not specifying whether linear or shear and this represents rotation ok.

So, the generalized deformation tensor can be decomposed into two parts. What is the first part? See this first part is symmetric because if you swap i and j it remains the same whereas, the second part is skew symmetric because if you swap i and j it becomes
negative of this, and this is actually a generalization of the matrix algebra rule that every matrix can be decomposed into a symmetric and a skew symmetric matrix. And here because there are two indices, we can actually write it in the form of a matrix, but tensor is something which is much more general than a matrix representation matrix is just a representation of a second order tensor. It becomes very tempting you know to bring in the perspective of the continuity equation, because this divergence of the velocity vector has already appeared.

So, we have clearly two choices left; one is we discuss the continuity equation as a separate topic as a separate chapter or because we already discussed volumetric deformation and volumetric deformation in some way is related to mass conservation we will see how. Therefore, we can bring in perspectives of mass conservation here. Mass conservation is not fluid kinematics that needs to be clarified but we are bringing that in the context of fluid kinematics because volume deformation is a aspect of fluid kinematics and mass conservation can be related to volume change. So, that is the whole picture that needs to be clarified before we discuss this.

So, the holy grail of mass conservation in fluid mechanics is the continuity equation. No matter how simple or how complex the flow is, if it does not satisfy continuity equation; that means you have a question right whether that kind of flow field will exist or not. So, being a fundamental equation it is a good intellectual exercise to make an attempt to derive the continuity equation from various considerations.

So, in this particular course, I will try to have three different methods of deriving the continuity equation and we will show that how all these converge to the same outcome. So, the first approach is.
So, we are talking about the continuity equation control mass approach. So, in control mass approach essentially you are talking about a fixed mass and that mass is the density times the volume. We write the volume in this way because the velocity is also V just to distinguish the symbol. So, you can take log of both sides and write this as.

Now, what we do is, we take or we differentiate both sides, but what we will do is we will cleverly switch from the Lagrangian variable to Eulerian variable and that we can do if instead of the ordinary derivative we put the total derivative. So, ordinary derivative to total derivative is a paradigm shift for transferring from Lagrangian to Eulerian approach.

So, you can write; because mass of a system is conserved this is the law of conservation of mass. So, this must be 0, what is this? This is the volumetric strain. So, this is the divergence of the velocity vector. What is this? This is the total derivative operator applied over rho.

So, if you combine these two together that is you know multiply both, multiply these by rho. So, then right; so, V dot nabla rho plus rho into nabla dot V. So, that is nabla dot rho V, right. So, this is the continuity equation the general form of the continuity equation in vector form.
So, you see that how from the volume conservation or you know not really volume conservation the volume deformation consideration, we can quickly come up with a mass conservation equation using the total derivative as the notation that we have already learned.

The other more traditional approach which we normally follow in all classes is the control volume approach.

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So, for this we assume that there is a rectangular control volume. So, that we can use a Cartesian system for our derivation, again this box has dimensions of delta x.

So, what we will write is simply an expression of mass conservation for these. So, I will just make it little bit quick. So, this is \( m \dot{x} \) the mass flow rate at \( x \), this is mass flow rate at \( x + \delta x \). Similarly you have \( m \dot{y} \), this is \( m \dot{y} + \delta y \) from the backside you have \( m \dot{z} \) and from the front side you have \( m \dot{z} + \delta z \). So, you can write \( m \dot{x} + \delta x \) as \( m \dot{x} + \) higher order terms.

So, \( m \dot{x} + \delta x - m \dot{x} \) is what? Why we are calculating this? This is out minus in or in fact, you can calculate the other way \( m \dot{x} - m \dot{x} + \delta x \).
So, whatever is going in minus whatever is coming out. That is more rational to calculate because mass is not created whatever is coming in what we expect is that the lower than that will come out at the most that can come out, but not more than that.

But you know it is possible that this itself can be positive negative or 0 because it all depends on what are the mass flow rates in the other directions also that needs to be carefully considered. But so, this is similar expressions you can write for y and z.

So, you can write for the mass balance, the mass momentum energy whatever can be expressed by a balance law, in minus out plus generated equal to change. So, in minus out there is no mass generated is equal to the rate of change of mass within the control volume so that means, m dot x minus m dot x plus delta x plus m dot y minus m dot y plus delta y plus m dot z minus m dot z plus delta z.

What is m dot cv? Rho into delta x into delta y into delta z. So, here we calculate just one term, that is minus del del x of m dot x into delta x and this m dot x is nothing but rho into u into delta y into delta z. So, if you now combine all these terms and take the limit as delta x delta y delta z tends to 0; why you take this limit? This is the more important question than doing the algebra. You take this
limit because you want to derive a differential equation. Differential equation is an equation which is valid at a point. So, you must shrink the volume to the point. So, that is why you take the limit.

So, when you take the limit as delta x, delta y, delta z all tending to 0 and also delta t, then you are left with this equation only because del del x of rho u plus del del y of rho v plus del del z of rho w summarily is divergence of rho into v. So, the same equation we have seen how it can be derived from control mass and control volume approach.

The next concept with which we will wind up today is the most important concept that what is special about incompressible flow. So, we have a notion in our mind and it is a mental block or prejudice that incompressible flow means density equal to constant.

Let us look into that more carefully. So, incompressible flow that special case we will discuss for about 2 minutes or so and then we will call it a day.

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So, incompressible flow; we already know that for incompressible flow divergence of the velocity vector is equal to 0 this is derived from pure kinematic constraints.

So, if you recall the control mass based analysis then what was it; D rho D t, right control mass based equation. So, if divergence of the velocity vector is 0; if this is 0, then this will be automatically 0 because sum total of this is 0. So, incompressible flow also means total derivative of density equal to 0.
So, in a Cartesian coordinate system it means what? This does not necessarily mean that rho is constant. You may have a variable rho, but these terms get adjusted with each other in such a way that sum total is 0. So, clearly incompressible flow does not necessarily mean that density is a constant. If density is a constant this equation is trivially satisfied. So, density equal to constant is a special type of incompressible flow, but with variable density also you can have incompressible flow.

I will give you a simple algebraic example. Let us say you have rho equal to kxy. Can you define an incompressible flow for this? So, to do that, so, question is can it represent an incompressible flow? So, to give an answer to this we will calculate this total derivative. So, assuming that is this is not a function of time. So, it is a two dimensional case. So, it is just whether this can be satisfied. So, u del rho del x is uky plus v del rho del y is equal to 0. So, u by v is equal to minus x by y.

So, you can have a flow field with u equal to cx and v is equal to minus cy ok. So, this satisfies the divergence of the velocity 0, but you can have a density which is a function of both x and y. So, physically it may be possible because there is a volume and it is possible that the advection that is the flow across all the faces plus the transient effect together maybe this is because of change of face that there is a density change, sum total of these may come out to be 0 and that we still call as incomprehensible flow although density is not a constant.

With this little bit of note and I believe this is a very important note, because most students have a feel that incompressible flow must be a constant density flow; eventually it is not. Let us stop here today and we will continue in the next lecture.

Thank you.