Problem 1: Machine precision is defined as the largest number $\epsilon$ such that $1 + \epsilon = 1$. Write a C program to calculate $\epsilon$. Calculate it for both 'float' and 'double' data types.

Problem 2: a) Write a C program to evaluate the function $e^{-x}$ at $x = 0.5$ by summing the series:

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots$$

Sum the series until $|\text{term sum}| < 10^{-8}$.
b) Calculate the relative error by comparing with the exact value obtained from in-built library function 'exp'.
c) Plot (log-log) relative error versus number of terms kept in the series.

Problem 3: We know that roots of a quadratic equation of the form $ax^2 + bx + c = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Write a C program to evaluate the roots of such a quadratic equation with $a = 1, b = 3000.001, c = 3$.

Actual roots are $x_1 = -0.001$ and $x_2 = -3000$. Use single precision ('float') to represent real numbers.

Problem 4: The $n^{th}$ power of the number $\phi = \frac{\sqrt{5} - 1}{2} \approx 0.61803398$ obeys a recursion relation

$$\phi^{n+1} = \phi^{n-1} - \phi^n$$

Starting with $\phi^0 = 1$ and $\phi^1 = 0.61803398$ evaluate the values upto $n = 20$ using the above recursion relation. Compare your results with $\phi^n$ evaluated using the in-built 'pow' function. Plot the relative error as a function of $n$.

Problem 5: If divided differences are defined as

$$f(x, x_0) = \frac{y - y_0}{x - x_0}$$

$$f(x, x_0, x_1) = \frac{f(x, x_0) - f(x_0, x_1)}{x - x_1}$$

$$f(x, x_0, x_1, x_2) = \frac{f(x, x_0, x_1) - f(x_0, x_1, x_2)}{x - x_2}$$

a) prove that divided differences has the following symmetries with respect to exchange of arguments

$$f(x_1, x_0) = f(x_0, x_1)$$

$$f(x_2, x_1, x_0) = f(x_2, x_0, x_1)$$

$$f(x_2, x_1, x_0) = f(x_1, x_2, x_0)$$

$$f(x_3, x_2, x_1, x_0) = f(x_2, x_3, x_1, x_0)$$
Problem 6: Write a C program to perform Newton’s interpolation (Eq. 1) and find $\log_{10} 323.5$ using the following table.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\log_{10} x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>321.0</td>
<td>2.50651</td>
</tr>
<tr>
<td>322.0</td>
<td>2.50893</td>
</tr>
<tr>
<td>324.2</td>
<td>2.51081</td>
</tr>
<tr>
<td>325.0</td>
<td>2.51188</td>
</tr>
</tbody>
</table>

Problem 7: If $y = P(x)$ is a polynomial of $n^{th}$ degree which takes the values $y_0, y_1, y_2, ..., y_n$ when $x$ has the values $x_0, x_1, x_2, ..., x_n$ respectively and $(n + 1)^{th}$ divided differences of this polynomial is given as

$$f(x_0, x_1, x_2, ..., x_n) = \frac{y}{(x - x_0)(x - x_1)(x - x_2)\cdots(x - x_n)}$$

$$+ \frac{y_0}{(x - x_1)(x - x_2)\cdots(x - x_n)}$$

$$+ \frac{y_1}{(x - x_2)\cdots(x - x_n)}$$

$$+ \cdots$$

$$+ \frac{y_n}{(x - x_n)(x - x_{n-1})}$$

Show that $y$ can be written as

$$y = \frac{(x - x_1)(x - x_2)\cdots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\cdots(x_0 - x_n)}y_0$$

$$+ \frac{(x - x_0)(x - x_2)\cdots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)\cdots(x_1 - x_n)}y_1$$

$$+ \frac{(x - x_0)(x - x_1)(x - x_3)\cdots(x - x_n)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)\cdots(x_2 - x_n)}y_2$$

$$+ \cdots$$

$$+ \frac{(x - x_0)(x - x_1)(x - x_2)\cdots(x - x_{n-1})}{(x_n - x_0)(x_n - x_1)(x_n - x_3)\cdots(x_n - x_{n-1})}y_n$$

which is Lagrange’s interpolation formula.

Hint: You can use the fact that $n^{th}$ divided difference of a polynomial of degree $n$ is constant. Hence $(n + 1)^{th}$ divided difference of a polynomial of degree $n$ is zero.
• Problem 8: a) Implement a C function to perform Gauss Elimination  
b) Use it to solve  

\[
\begin{align*}
0.0003x_1 + 3.0000x_2 &= 2.0001 \\
1.0000x_1 + 1.0000x_2 &= 1.0000
\end{align*}
\]
Use 'float' datatype for all real numbers. The exact solution is \(x_1 = \frac{1}{3}, x_2 = \frac{2}{3}\)  
c) Solve the above problem with pivoting.

• Problem 9: a) Extend the Gauss Elimination function to calculate LU decomposition of a matrix.  
b) Using LU decomposition, solve the system of equations:  

\[
\begin{align*}
7x_1 + 2x_2 - 3x_3 &= -12 \\
2x_1 + 5x_2 - 3x_3 &= -20 \\
x_1 - x_2 - 6x_3 &= -26
\end{align*}
\]

• Problem 10: a) Solve the following set of equations by LU decomposition without pivoting  

\[
\begin{align*}
x_1 + 7x_2 - 4x_3 &= -51 \\
4x_1 - 4x_2 + 9x_3 &= 62 \\
12x_1 - x_2 + 3x_3 &= 8
\end{align*}
\]
b) Determine the matrix inverse. Check whether \([A][A]^{-1} = [I]\)