1. Given data: $N = 4$, $L = 2$, $\sigma_h^2 = 1/4$, $\sigma^2 = 1/2$, minimum cyclic prefix (CP) = $L-1 = 1$.

The pilot symbols transmitted on all the subcarriers are given as $X(0) = 3 + 3j$, $X(1) = -2 - 2j$, $X(2) = -1 + j$, $X(3) = 2 - j$ and the received samples in the time domain be $y(0) = -1 - j$, $y(1) = 2 + 2j$, $y(2) = 3 - 2j$, $y(3) = 3 - 2j$. Time domain sample is generated by $N = 4$ pt IDFT and the $k$th sample is given by,

$$x(k) = \frac{1}{N} \sum_{l=0}^{N-1} X(l)e^{j2\pi \frac{k l}{N}}.$$ 

The time domain samples $x(0)$, $x(1)$, $x(2)$ and $x(3)$ can be calculated
as,
\[
x(0) = \frac{1}{4} \sum_{l=0}^{3} X(l) e^{j\frac{\pi}{2} l} \]
\[
= \frac{1}{4} \sum_{l=0}^{3} X(l) \\
= \frac{1}{2} + \frac{1}{4}j. \\
x(1) = \frac{1}{4} \sum_{l=0}^{3} X(l) e^{j\frac{\pi}{2} l} \\
= \frac{1}{4} \left\{ 3 + 3j + j(-2 - 2j) - (-1 + j) - j(2 - j) \right\} \\
= \frac{5}{4} - \frac{1}{2}j. \\
x(2) = \frac{1}{4} \sum_{l=0}^{3} X(l) e^{j\frac{\pi}{2} l} \\
= \frac{1}{4} \left\{ 3 + 3j - (-2 - 2j) + (-1 + j) - (2 - j) \right\} \\
= \frac{1}{2} + \frac{7}{4}j. \\
x(3) = \frac{1}{4} \sum_{l=0}^{3} X(l) e^{j\frac{3\pi}{2} l} \\
= \frac{1}{4} \left\{ 3 + 3j - j(-2 - 2j) - (-1 + j) + j(2 - j) \right\} \\
= \frac{3}{4} + \frac{3}{2}j. \\
\]
The transmitted block of samples in the time domain is given by,
\[
x(3), x(0), x(1), x(2), x(3) \\
= \frac{3}{4} + \frac{3}{2}j, \frac{1}{2} + \frac{1}{4}j, \frac{5}{4} - \frac{1}{2}j, \frac{1}{2} + \frac{7}{4}j, \frac{3}{4} + \frac{3}{2}j. \\
\]
\text{Ans (c)}

2. The received symbols across the subcarriers are generated by \( N = 4 \) pt FFT and the \( k \)th sample is given by,
\[
Y(l) = \sum_{k=0}^{N-1} y(k)e^{-j2\pi\frac{kl}{N}}. \\
\]
\[2\]
So, the received samples $Y(0)$, $Y(1)$, $Y(2)$ and $Y(3)$ can be calculated as,

\[
Y(0) = \sum_{k=0}^{3} y(k)e^{-j\frac{\pi}{2}0} \\
= \sum_{k=0}^{3} y(k) \\
= 7 - 3j,
\]

\[
Y(1) = \sum_{k=0}^{3} y(k)e^{-j\frac{\pi}{2}k} \\
= -1 - j - j(2 + 2j) - (3 - 2j) + j(3 - 2j) \\
= 2j,
\]

\[
Y(2) = \sum_{k=0}^{3} y(k)e^{-j\pi k} \\
= -1 - j - (2 + 2j) + 3 - 2j - (3 - 2j) \\
= -3 - 3j,
\]

\[
Y(3) = \sum_{k=0}^{3} y(k)e^{-j\frac{3\pi}{2}k} \\
= -1 - j + j(2 + 2j) - (3 - 2j) - j(3 - 2j) \\
= -8.
\]

The received symbols across the subcarriers are,

$7 - 3j, 2j, -3 - 3j, -8$.

**Ans (b)**

3. The system model is given by,

\[
Y(l) = H(l)X(l) + V(l).
\]

The LMMSE estimate of $H(l)$ is given by,

\[
\hat{H}(l) = \frac{L\sigma_h^2 X^*(l)}{L\sigma_h^2 |X(l)|^2 + N\sigma^2 Y(l)}.
\]
The LMMSE estimates $\hat{H}(0)$, $\hat{H}(1)$ of the channel coefficients $H(0)$, $H(1)$ across subcarriers 0, 1 respectively can be calculated as,

$$
\hat{H}(0) = \frac{(3 - 3j)(7 - 3j)}{22} = \frac{6}{11} - \frac{15}{11}j;
$$

$$
\hat{H}(1) = \frac{(-2 + 2j)(2j)}{12} = -\frac{1}{3} - \frac{1}{3}j.
$$

**Ans (d)**

4. The system model is given by,

$$
Y(l) = H(l)X(l) + V(l).
$$

The LMMSE estimate of $H(l)$ is given by,

$$
\hat{H}(l) = \frac{L\sigma^2_h X^*(l)}{L\sigma^2_h |X(l)|^2 + N\sigma^2} Y(l).
$$

The LMMSE estimates $\hat{H}(2)$, $\hat{H}(3)$ of the channel coefficients $H(2)$, $H(3)$ across subcarriers 2, 3 respectively can be calculated as,

$$
\hat{H}(2) = \frac{(-1 - j)(-3 - 3j)}{6} = j;
$$

$$
\hat{H}(3) = \frac{(2 + j)(-8)}{9} = -\frac{16}{9} - \frac{8}{9}j.
$$

**Ans (a)**

5. The LMMSE estimate of $H(l)$ is given by,

$$
\hat{H}(l) = \frac{L\sigma^2_h X^*(l)}{L\sigma^2_h |X(l)|^2 + N\sigma^2} Y(l).
$$

At high SNR, i.e $L\sigma^2_h |X(l)|^2 >> N\sigma^2$. The ML estimate of $H(l)$ is given by,

$$
\hat{H}(l) = \frac{1}{X(l)} Y(l).
$$
The ML estimates $\hat{H}(2), \hat{H}(3)$ of the channel coefficients $H(2), H(3)$ across subcarriers 2, 3 respectively can be calculated as,

$$
\hat{H}(2) = \frac{-3 - 3j}{-1 + j} = 3j,
$$
$$
\hat{H}(3) = \frac{-8}{2 - j} = -\frac{16}{5} - \frac{8}{5}j.
$$

Ans (b)

6. The MSE of the LMMSE estimate is given by,

$$
E\{|\hat{H}(l) - H(l)|^2\} = \frac{1}{N\sigma^2/|X(l)|^2 + \frac{1}{L\sigma_h^2}}.
$$

The MSEs of the LMMSE estimates $\hat{H}(0), \hat{H}(1), \hat{H}(2), \hat{H}(3)$ of the channel coefficients $H(0), H(1), H(2), H(3)$ across subcarriers 0, 1, 2, 3 respectively can be calculated as

$$
E\{|\hat{H}(0) - H(0)|^2\} = \frac{1}{\frac{1}{4\sigma^2/|X(0)|^2} + \frac{1}{2\sigma_h^2}} = \frac{1}{11},
$$
$$
E\{|\hat{H}(1) - H(1)|^2\} = \frac{1}{\frac{1}{4\sigma^2/|X(1)|^2} + \frac{1}{2\sigma_h^2}} = \frac{1}{6},
$$
$$
E\{|\hat{H}(2) - H(2)|^2\} = \frac{1}{\frac{1}{4\sigma^2/|X(2)|^2} + \frac{1}{2\sigma_h^2}} = \frac{1}{3},
$$
$$
E\{|\hat{H}(3) - H(3)|^2\} = \frac{1}{\frac{1}{4\sigma^2/|X(3)|^2} + \frac{1}{2\sigma_h^2}} = \frac{2}{9}.
$$

Ans (a)
7. The noise sample $V(l)$ on the $l$th subcarrier is given by,

$$V(l) = \sum_{k=0}^{N-1} v(k)e^{-j2\pi \frac{kl}{N}},$$

where mean of $V(l)$ is given by,

$$E\{V(l)\} = E\left\{ \sum_{k=0}^{N-1} v(k)e^{-j2\pi \frac{kl}{N}} \right\}$$

$$= \sum_{k=0}^{N-1} E\{v(k)\}e^{-j2\pi \frac{kl}{N}}$$

$$= 0,$$

and variance of $V(l)$ is given by,

$$E\{|V(l)|^2\} = E\{V(l)V^*(l)\}$$

$$= E\left\{ \left( \sum_{k=0}^{N-1} v(k)e^{-j2\pi \frac{kl}{N}} \right) \left( \sum_{\tilde{k}=0}^{N-1} \tilde{v}(\tilde{k})e^{-j2\pi \frac{\tilde{k}l}{N}} \right) \right\}$$

$$= N\sigma^2$$

$$= 2.$$

So, the noise sample $V(l)$ on the $l$th subcarrier is zero-mean Gaussian with variance 2.

**Ans (d)**

8. The channel coefficient $H(l)$ on the $l$th subcarrier is given by,

$$H(l) = \sum_{k=0}^{N-1} h(k)e^{-j2\pi \frac{kl}{N}},$$

where mean of $V(l)$ is given by,

$$E\{H(l)\} = E\left\{ \sum_{k=0}^{L-1} h(k)e^{-j2\pi \frac{kl}{N}} \right\}$$

$$= \sum_{k=0}^{L-1} E\{h(k)\}e^{-j2\pi \frac{kl}{N}}$$

$$= 0,$$
and variance of $V(l)$ is given by,

$$E\{|H(l)|^2\} = E\{H(l)H^*(l)\}$$

$$= E\left\{ \left( \sum_{k=0}^{L-1} h(k)e^{-j2\pi \frac{k}{N}} \right) \left( \sum_{\tilde{k}=0}^{L-1} \tilde{h}(\tilde{k})e^{-j2\pi \frac{\tilde{k}l}{N}} \right) \right\}$$

$$= L\sigma_h^2$$

$$= \frac{1}{2}.$$

So, the channel coefficient $H(l)$ on the $l$th subcarrier is zero-mean with variance $\frac{1}{2}$.

**Ans (b)**

9. The LMMSE estimates of the channel tap $\hat{h}(k)$ is given by,

$$\hat{h}(k) = \frac{1}{N} \sum_{l=0}^{N-1} \hat{H}(l)e^{j2\pi \frac{kl}{N}}.$$

The LMMSE estimates $\hat{H}(0), \hat{H}(1), \hat{H}(2), \hat{H}(3)$ of the channel coefficients $H(0), H(1), H(2), H(3)$ across subcarriers 0, 1, 2 and 3 respectively are,

$$\hat{H}(0) = \frac{(3 - 3j)(7 - 3j)}{22}$$

$$= \frac{6}{11} - \frac{15}{11}j,$$

$$\hat{H}(1) = \frac{(-2 + 2j)(2j)}{12}$$

$$= \frac{1}{3} - \frac{1}{3}j,$$

$$\hat{H}(2) = \frac{(-1 - j)(-3 - 3j)}{6}$$

$$= j,$$

$$\hat{H}(3) = \frac{(2 + j)(-8)}{9}$$

$$= -\frac{16}{9} - \frac{8}{9}j.$$

The LMMSE estimates of the channel taps $\hat{h}(0), \hat{h}(1)$ respectively can
be calculated as,

\[
\hat{h}(0) = \frac{1}{4} \sum_{l=0}^{3} \hat{H}(l)e^{j\frac{\pi}{2}l}
\]

\[
= \frac{1}{4} \left\{ \frac{6}{11} - \frac{15}{11}j - \frac{1}{3} - \frac{1}{3}j + j - \frac{16}{9} - \frac{8}{9}j \right\}
\]

\[
= \frac{155}{396} - \frac{157}{396}j
\]

\[
\hat{h}(1) = \frac{1}{4} \sum_{l=0}^{3} \hat{H}(l)e^{j\frac{\pi}{2}l}
\]

\[
= \frac{1}{4} \left\{ \frac{6}{11} - \frac{15}{11}j + j \left( -\frac{1}{3} - \frac{1}{3}j \right) - j \left( -\frac{16}{9} - \frac{8}{9}j \right) \right\}
\]

\[
= \frac{1}{396} - \frac{91}{396}j.
\]

Ans (c)

10. As seen from solution to 6 above, variance in estimate of \(H(1)\) is \(\frac{1}{6}\).

As we know, since the pilot block is repeated \(M\) times, the variance corresponding to \(M\) pilots on each subcarrier is \(\frac{1}{M} = \frac{1}{6M}\). Let \(\hat{H}_R(1)\) denotes the real part of the estimate \(\hat{H}(1)\). Further, \(\hat{H}_R(1) - H_R(1)\) gives the estimation error in the real part of the estimate. Therefore, \(\hat{H}_R(1) - H_R(1)\) is distributed as a zero-mean Gaussian with variance \(\frac{1}{12M}\). Therefore, \(\frac{\hat{H}_R(1) - H_R(1)}{\sqrt{\frac{1}{12M}}}\) is a zero-mean unit-variance Gaussian RV.

Therefore, probability that the real part of the MMSE estimate \(\hat{H}(1)\) lies within a radius \(1/8\) of the real part of the true parameter \(H(1)\) can be calculated as follows,

\[
\Pr \left( |\hat{H}_R(1) - H_R(1)| \leq \frac{1}{8} \right) = \Pr \left( \frac{|\hat{H}_R(1) - H_R(1)|}{\sqrt{\frac{1}{12M}}} \leq \frac{\frac{1}{8}}{\sqrt{\frac{1}{12M}}} \right)
\]

\[
= 1 - \Pr \left( \frac{|\hat{H}_R(1) - H_R(1)|}{\sqrt{\frac{1}{12M}}} \geq \frac{\frac{1}{8}}{\sqrt{\frac{1}{12M}}} \right)
\]

\[
= 1 - \left\{ \Pr \left( \frac{|\hat{H}_R(1) - H_R(1)|}{\sqrt{\frac{1}{12M}}} \geq \frac{\frac{1}{8}}{\sqrt{\frac{1}{12M}}} \right) + \Pr \left( \frac{|\hat{H}_R(1) - H_R(1)|}{\sqrt{\frac{1}{12M}}} \leq -\frac{\frac{1}{8}}{\sqrt{\frac{1}{12M}}} \right) \right\}
\]

\[
= 1 - 2\Pr \left( \frac{|\hat{H}_R(1) - H_R(1)|}{\sqrt{\frac{1}{12M}}} \geq \frac{\frac{1}{8}}{\sqrt{\frac{1}{12M}}} \right) = 1 - 2Q \left( \frac{\frac{1}{8}}{\sqrt{\frac{1}{12M}}} \right) = 1 - 2Q \left( \frac{\sqrt{\frac{12M}{64}}}{8} \right)
\]
Further, since the errors in the real and imaginary parts are independent as they are Gaussian, the probability that both the real and imaginary parts of the MMSE estimate $\hat{H}(1)$ lie within a radius of $1/8$ from the real and imaginary parts of the true parameter $H(1)$ respectively is

$$\left(1 - 2Q\left(\frac{\sqrt{12M/64}}{}\right)\right)^2.$$ 

Therefore, probability that both the real and imaginary parts of the MMSE estimate $\hat{H}(1)$ lie within a radius of $1/8$ from the real and imaginary parts of the true parameter $H(1)$ is greater than 99.99% is

$$\left(\Pr\left(|\hat{H}_R(1) - H_R(1)| \leq \frac{1}{8}\right)\right)^2 \geq .9999$$

$$1 - 2Q\left(\frac{\sqrt{12M/64}}{}\right) \geq .9999$$

$$2Q\left(\frac{\sqrt{12M/64}}{}\right) \leq 10^{-4}$$

$$Q\left(\frac{\sqrt{12M/64}}{}\right) \leq 5 \times 10^{-5}$$

$$\sqrt{12M/64} \geq Q^{-1}(5 \times 10^{-5})$$

$$M \geq \frac{16}{3}(Q^{-1}(5 \times 10^{-5}))^2$$

Ans (d)