1. a) Show that the Hamiltonian for the Central Field Potential is given by

\[ \hat{H} = \left( \frac{1}{2m} \right) \left( p_x^2 + \frac{\hbar^2 \ell^2}{r^2} \right) + U(r) \]

Where \( p, \psi = -i\hbar \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) \psi \)

b) If the solution to Schrodinger equation with above Hamiltonian for the special case of zero potential is written as

\[ \psi_{klm} = R_{kl}(r) Y_{l}(\theta, \phi) \]

Then show that the stationary state solutions satisfy the following normalization condition:

\[ \int_0^\infty r^2 R_{kl} r R_{k'l'} dr = 2\pi \delta(k' - k) \]

c) If the normalization is done on the energy scale instead of \( \left( \frac{k}{2\pi} \right) \) scale then show that

\[ R_{kl} = \left( 1 \right) \left( \frac{m}{2\pi \hbar^2 k} \right) R_{kl} \]

d) Prove that the energy normalization eigen function for a free particle in one dimension (along x axis) is given by

\[ \left( \frac{m}{8\pi^2 \hbar^2 E} \right)^{\frac{1}{2}} e^{\pm ikx} \]

2. a) Show that for a free particle \( R_{k0} = \frac{2\sin kr}{r} \)

b) \( R_{kl} = (-1)^l \frac{2r^l}{k^l} \left( \frac{1}{r} \frac{d}{dr} \right)^l \sin kr \)

c) \( R_{kl} = 2k j_l (kr) \)

where \( j_l (kr) \) are spherical Bessel functions.
3. Subject the solution to the Schrodinger equation for scattering of an electron by a central potential, given by (outgoing wave boundary conditions):
\[
\psi^+_{Tot} (\vec{r}, t) \mid_{r \to \infty} = e^{+i(kz+\alpha t)} + \frac{e^{+i(kr+\alpha t)}}{r} \left\{ \frac{1}{2ik} \sum_{l=0}^{\infty} (2l+1) \left[ e^{2i\delta_1(k)} - 1 \right] P_l \left( \cos \theta \right) \right\}
\]
To time-reversal symmetry and show clearly that the solution for photoelectron ejection is given by (ingoing wave boundary conditions):
\[
\psi^-_{Tot} (\vec{r}, t) \mid_{r \to \infty} = e^{+i(kz+\alpha t)} + \frac{e^{+i(kr+\alpha t)}}{r} \left\{ \frac{1}{2ik} \sum_{l=0}^{\infty} (2l+1) \left[ e^{2i\delta_1(k)} - 1 \right] P_l \left( \cos \theta \right) \right\}
\]

4. Obtain the phase shifts \( \delta \) produced by a repulsive potential \( V(r) = \frac{A}{r^2} \) with \( A > 0 \). What kind of angular distribution do you get? Is the scattering cross section finite?

5. If the scattering potential is attractive \( V(r) = \frac{A}{r^2} \) with \( A < 0 \), would the radial equation have solutions for all negative values or would there be any further restriction? Here, \( U(r) = \frac{2mV(r)}{\hbar^2} \) and \( u_i(k, r) = rR_i(k, r) \).

6. If the differential scattering cross section is written in the form
\[
\frac{d\sigma}{d\Omega} = A + BP_1(\cos \theta) + CP_2(\cos \theta)
\]
express the coefficients \( A, B & C \) in terms of the phase shifts \( \delta \).

7. a) Show that the general solution to the Schrodinger equation,
\[
\nabla^2 \psi + \frac{2m}{\hbar^2} \left[ E - V(r) \right] \psi = 0
\]
Can be written as \( \psi = \sum c_{im} \frac{F_i(\rho)}{\rho} Y_i^m(\theta) \) where \( F_i(\rho) \) is a solution of
\[
\frac{d^2F_i}{d\rho^2} + \left\{ 1 - \frac{V(r)}{E} - \frac{l(l+1)}{\rho^2} \right\} F_i = 0
\]

b) Prove that \( F_i(\rho \to \infty) = \sin \left( \rho - \frac{l\pi}{2} + \delta_i \right) \)
1. Prove that the scattering amplitude \( f(\theta) \) given in the asymptotic form

\[
\psi(r \to \infty) = e^{ikr} + f(\theta) \frac{e^{ikr}}{r}
\]

Is related to the differential cross section \( \frac{d\sigma}{d\Omega} \) by the following relation:

\[
\frac{d\sigma}{d\Omega} = |f(k, \theta, \phi)|^2
\]

2. Prove that the total scattering cross-section is given by \( \sigma_{\text{tot}} = \left( \frac{4\pi}{k} \right) \text{Im} f(\theta = 0) \).

References:

5. E. Merzbacker, Quantum Mechanics; Wiley International Edition. 1970