① Eqn of rocket motion,

$$\vec{F}_{ext} = M\frac{dv}{dt} - (u - v)\frac{dM}{dt}$$

M = Total mass of rocket = rocket + fuel $= 50 + 450 = 500\,kg$

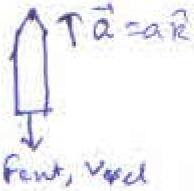
$\vec{F}_{ext} = mg$ = Ext. force on M.

$\vec{v}$ = vel. of mass M at any instant t.

$\vec{u}$ = vel. of burnt fuel (expelled mass)

$\vec{u} - \vec{v}$ = vel. of ejected gas w.r.t rocket $= \vec{v}_{rel}$

$$\vec{F}_{ext} = mg(-\hat{k}), \quad \vec{v}_{rel} = v_{rel}(-\hat{k}), \quad \vec{a} = \frac{d\vec{v}}{dt} = a(\hat{k})$$



$$\therefore \; Mg(-\hat{k}) = Ma(\hat{k}) - v_{rel}(-\hat{k})\frac{dM}{dt}$$

$$\Rightarrow -Mg - Ma = v_{rel}\frac{dM}{dt}$$

$$\Rightarrow -\frac{dM}{dt} = \frac{M(a+g)}{v_{rel}} = \frac{500 \times (20 + 9.8)}{2000}$$

$$\Rightarrow \boxed{-\frac{dM}{dt} = 7.45\ kg/s}$$

(i) $M\frac{dv}{dt} = -mg - v_{rel}\frac{dM}{dt}$

M_0 = Initial rocket mass = Empty rocket mass + Fuel mass
$= M_e + M_f$

$\frac{dM}{dt} = -\alpha M_0$

$\boxed{\beta = \frac{M_f}{M_0} = \text{Propelled Mass}}$

$M\frac{dv}{dt} = -Mg - v_{rel}\frac{dM}{dt}$

$\Rightarrow M\frac{dv}{dM}\frac{dM}{dt} = -Mg - v_{rel}\frac{dM}{dt}$

$\Rightarrow M\frac{dv}{dM}(-\alpha M_0) = -Mg - v_{rel}(-\alpha M_0)$

$\Rightarrow \frac{dv}{dM} = +\frac{g}{\alpha M_0} - \frac{v_{rel}}{M}$

$\Rightarrow \int_0^{v_f} dv = \frac{g}{\alpha M_0}\int_{M_0}^{M_e} dm - v_{rel}\int_{M_0}^{M_e}\frac{dM}{M_0}$

$\Rightarrow v_f = \frac{g}{\alpha M_0}(M_e - M_0) - v_{rel}\ln\frac{M_e}{M_0}$

$\Rightarrow v_f = -\frac{g}{\alpha M_0}(M_0 - M_e) + v_{rel}\ln\frac{M_0}{M_e}$

$M_0 = M_e + M_f$
$M_e << M_f$
$\therefore M_0 \approx M_f$
$M_0 - M_e \approx M_f$

$\Rightarrow v_f \approx -\frac{g}{\alpha M_0}M_f + v_{rel}\ln\frac{M_f}{M_e} = -\frac{g}{\alpha} + v_{rel}\ln\frac{M_f}{M_e}$

$\Rightarrow \frac{M_f}{M_e} \approx \exp\left(\frac{v_f + g/\alpha}{v_{rel}}\right) = \exp\left[\frac{11200 + (60\times 9.8)}{2100}\right]$
$= \exp(5.613) = 274.05$

$\therefore \boxed{\frac{M_f}{M_e} \approx 275}$

(3) Both (ii) and (iii) are true

(4) $E >, <$ and $= 0$ correspond to the path ~~parabo~~ is a hyperbola, an ellipse and parabola respectively.

(5) Rocket equ^n,

$$m\frac{dv}{dt} + u\frac{dm}{dt} = -mg$$

For minimum exhaust velocity, $\frac{dv}{dt} = 0$

$$\therefore\ u\frac{dm}{dt} = -mg \Rightarrow -dm$$

$$\Rightarrow u = -\frac{mg}{\frac{dm}{dt}} = +\frac{mg}{\alpha}$$

$$\Rightarrow \boxed{u = \frac{mg}{\alpha}}$$ $\Rightarrow$ Minimum exhaust velocity.

(6) $m\frac{dv}{dt} + v\frac{dm}{dt} = mg \Rightarrow \frac{dv}{dt} + \frac{v}{\frac{4}{3}\pi r^3\rho}\frac{d}{dt}\left(\frac{4}{3}\pi r^3\rho\right) = g$

$$\boxed{\begin{aligned} r &= \frac{k}{\rho}t \\ \Rightarrow \frac{dr}{dt} &= k/\rho \end{aligned}}$$

$\Rightarrow \frac{dv}{dt} + \frac{v}{r^3\rho}\cdot 3r^2\cdot\frac{dr}{dt} = g \Rightarrow \frac{dv}{dt} + \frac{3v}{r}\frac{k}{\rho} = g$

$\Rightarrow \frac{dv}{dt} + \frac{3}{t}v = g$ $\qquad$ I.F. $= e^{\int\frac{3}{t}dt} = e^{\ln t^3} = t^3$

$\Rightarrow t^3\frac{dv}{dt} + 3vt^2 = gt^3$ [multiplying both sides by I.F.]

$\Rightarrow \frac{d}{dt}(vt^3) = gt^3$

$\Rightarrow \int d(vt^3) = g\int t^3 dt + c$

$\Rightarrow vt^3 = \frac{g}{4}t^4 + c$ $\qquad$ At, $t = 0$, $V = 0$ $\quad \therefore c = 0$

$\therefore vt^3 = \frac{g}{4}t^4$

$\Rightarrow \boxed{V = \frac{1}{4}gt}$

⑦ $\vec{L} = \vec{r} \times \vec{p}$

$\Rightarrow \frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = \underbrace{\vec{v} \times m\vec{v}}_{=0} + \vec{r} \times \vec{F}(r)$

$\Rightarrow \frac{d\vec{L}}{dt} = \vec{r} \times F(r)\hat{r}$ ($\because$ under central force field)

$\Rightarrow \frac{d\vec{L}}{dt} = 0 \Rightarrow$ Does not change in time

(8) $\vec{F} = \frac{r(r-1)}{r^2+1}\hat{r}$

for, $0 < r < 1$, $\vec{F} = -ve \Rightarrow$ attractive.

and $r > 1$, $\vec{F} = +ve \Rightarrow$ Repulsive.

(9) $q\vec{E}$ is the central force field.

As $\vec{E} = -\frac{q}{r^2}\hat{r}$

(10) $\vec{F} = \left(\frac{\alpha}{r^2} + \frac{\beta}{r^3}\right)\hat{r} = -\vec{\nabla}V = -\frac{\partial V}{\partial r}\hat{r}$

$\therefore V = -\int \vec{F}\cdot d\vec{r} = -\alpha\int\frac{dr}{r^2} - \beta\int\frac{dr}{r^3}$

$= -\alpha\cdot\frac{r^{-2+1}}{-2+1} - \beta\,\frac{r^{-3+1}}{-3+1} = \frac{\alpha}{r} + \frac{\beta}{2r^2}$

$\Rightarrow \boxed{V(r) = \frac{\alpha}{r} + \beta/2r^2}$