1. (b) have all elements real and positive.

2. \( v(x) = x^4 - 4x^3 - 8x^2 + 48x \)

For any equilibrium point, \( \frac{dv}{dx} > 0 \)

\( \Rightarrow 4x^3 - 12x^2 - 16x + 48 = 0 \)

\( \Rightarrow x^3 - 3x^2 - 4x + 12 = 0 \)

\( \Rightarrow (x-3)(x^2+4) = 0 \Rightarrow x = 3, \pm 2 \)

\( \frac{d^2v}{dx^2} = 12x^2 - 24x - 16 \)

\( \frac{d^2v}{dx^2} \bigg|_{x=2} = 48 - 48 - 16 = -16 < 0 \Rightarrow \text{unstable equilibrium} \)

\( \frac{d^2v}{dx^2} \bigg|_{x=-2} = 48 + 48 - 16 = 80 > 0 \Rightarrow \text{stable equilibrium} \)

\( \frac{d^2v}{dx^2} \bigg|_{x=3} = 108 - 72 - 16 = 20 > 0 \Rightarrow \text{stable equilibrium} \)

\( \Rightarrow \text{stable at } x = -2, 3, \text{ unstable at } x = 2. \)

3. (d) Fig. d does not represent normal modes.

As in this case, the molecule is translating as a whole downward, its centre of mass is not fixed.

4. \( x_1 = l \sin \theta_1 \)

\( x_2 = l \sin \theta_1 + l \sin \theta_2 \)

\( y_1 = l (1 - \cos \theta_1) \)

\( y_2 = l (1 - \cos \theta_1) + l (1 - \cos \theta_2) \)
\[ \dot{x}_1 = l \cos \theta_1 \dot{\theta}_1, \quad \dot{x}_2 = l (\cos \theta_1 \dot{\theta}_1 + \cos \theta_2 \dot{\theta}_2) \]
\[ \dot{y}_1 = l \sin \theta_1 \dot{\theta}_1, \quad \dot{y}_2 = l (\sin \theta_1 \dot{\theta}_1 + \sin \theta_2 \dot{\theta}_2) \]
\[ T = \frac{1}{2} m \left( \dot{x}_1^2 + \dot{y}_1^2 \right) + \frac{1}{2} m \left( \dot{x}_2^2 + \dot{y}_2^2 \right) \]
\[ = \frac{1}{2} ml^2 \left( \dot{\theta}_1^2 (\cos \theta_1 + \sin \theta_1) + \dot{\theta}_2^2 (\cos \theta_2 + \sin \theta_2) \right. \]
\[ + \dot{\theta}_2^2 (\cos \theta_2 + \sin \theta_2) + 2 \dot{\theta}_1 \dot{\theta}_2 \cos (\theta_1 - \theta_2) \]
\[ = \frac{1}{2} ml^2 \left( 2 \dot{\theta}_1^2 + \dot{\theta}_2^2 + 2 \dot{\theta}_1 \dot{\theta}_2 \right) \quad \text{[for small angle approx. \( \cos \theta \approx 1 \)]} \]
\[ 2T = \left( \begin{array}{c} \dot{\theta}_1 \\ \dot{\theta}_2 \end{array} \right) \left( \begin{array}{cc} 2ml^2 & ml^2 \\ ml^2 & ml^2 \end{array} \right) \left( \begin{array}{c} \dot{\theta}_1 \\ \dot{\theta}_2 \end{array} \right) \]
\[ V = mg (y_1 + y_2) = mg l \left[ (1 - \cos \theta_1) + (1 - \cos \theta_2) \right. \]
\[ = mg l \left[ (1 - 1 + \theta_1^2) + (1 - 1 + \theta_2^2) \right] \]
\[ 2V = mg l \left( \begin{array}{c} \theta_1 \\ \theta_2 \end{array} \right) \left( \begin{array}{cc} 2ml^2 & 0 \\ 0 & 2ml^2 \end{array} \right) \left( \begin{array}{c} \theta_1 \\ \theta_2 \end{array} \right) \]

For non-trivial soln. of normal modes,

\[ |V - W| < 0 \]
\[ \Rightarrow \begin{vmatrix} 2ml^2 - 2mw^2 \xi^2 & -w^2 ml^2 \\ -w^2 ml^2 & ml^2 - mga \end{vmatrix} = 0 \]
\[ \Rightarrow \begin{vmatrix} 2g - 2lw^2 & -w^2 l \\ -w^2 l & g - w^2 l \end{vmatrix} = 0 \]
\[ \Rightarrow 2 (g - lw^2)^2 - w^4 l^2 = 0 \]
\(2g^2 - 4glw^2 + 2wl^2 - w^4 l^2 = 0\)

\(w^4 l^2 - 4glw^2 + 2g^2 = 0\)

\[w^2 = \frac{4gl \pm \sqrt{4g^2 - 4l^2 \cdot 2g^2}}{2l^2} = \frac{4gl \pm 2gl \sqrt{4 - 2}}{2l^2} = (2 \pm \sqrt{2}) \frac{g}{l}\]

\[w = \sqrt{(2 \pm \sqrt{2}) \frac{g}{l}}\]

\[T = \frac{1}{2} m q_1^2 + \frac{1}{2} m q_2^2\]

\[2T = \begin{pmatrix} q_1 & q_2 \end{pmatrix} \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}\]

\[V = \frac{1}{2} k_0 q_1^2 + \frac{1}{2} k_0 q_2^2 + \frac{1}{2} k_0 (q_2 - q_1)^2\]

\[2V = k_0 (2q_1^2 + 2q_2^2 + 2q_2 q_1)\]

\[= \begin{pmatrix} q_1 & q_2 \end{pmatrix} \begin{pmatrix} 2k_0 & 0 \\ 0 & 2k_0 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}\]

\[V = 2w^2 \bar{T} = 0\]

\[\begin{vmatrix} 2k_0 - w^2m & -k_0 \\ -k_0 & 2k_0 - mw^2 \end{vmatrix} = 0\]

\[(2k_0 - w^2m)^2 - k_0 = 0\]

\[2k_0 - w^2m = \frac{3k_0}{2}\]

\[w^2 = \frac{2k_0 + \sqrt{3k_0}}{m}\]

\[\lambda = \sqrt{\frac{k_0}{m}}, \sqrt{\frac{3k_0}{m}}\]
\[
\sum_j \left( Y_{ij} - \omega_k a_{jk} T_{ij} \right) a_{jk} = 0, \quad i = 1, 2, 3.
\]

\[
\begin{vmatrix}
2k_0 - m \omega_k & -k_0 \\
-k_0 & 2k_0 - m \omega_k
\end{vmatrix}
\begin{pmatrix}
a_{1k} \\
a_{2k}
\end{pmatrix} = 0
\]

For \( k = n = \sqrt{\frac{k_0}{2m}} \sqrt{x_0/m} \):

\[
k_0 a_{11} - k_0 a_{21} = 0 \quad \Rightarrow \quad a_{11} = a_{21}
\]

\[
-k_0 a_{11} + k_0 a_{21} = 0
\]

Normalization Condition:

\[
\sum_{ij} T_{ij} a_{jk} a_{jk} = \delta_{ik}
\]

\[
\sum_i a_{ik} = 1
\]

\[
m \left( a_{1k}^2 + a_{2k}^2 \right) = 1
\]

\[
\Rightarrow m(a_{11}^2 + a_{21}^2) = 1 \quad \Rightarrow a_{11} = a_{21} = \frac{1}{\sqrt{2m}}
\]

\[
a_1 = \frac{1}{\sqrt{2m}} (1)
\]

For \( \omega_k = \omega_2 = \sqrt{3x_0/m} \):

\[
k_0 a_{12} - k_0 a_{22} = 0 \quad \Rightarrow \quad a_{12} = -a_{22}
\]

\[
k_0 a_{12} = k_0 a_{22} = 0
\]

Similarly,

\[
a_2 = \frac{1}{\sqrt{2m}} (1)
\]
$$K.E.T = \frac{1}{2} m \dot{q}_1^2 + \frac{1}{2} m \dot{q}_2^2$$

$$2T = (\dot{q}_1, \dot{q}_2) (0 \text{ } m \text{ } \text{ } 0) (\dot{q}_1, \dot{q}_2)$$

P.E. $V = \frac{1}{2} K q_1^2 + \frac{1}{2} K q_2^2 + \frac{1}{2} A q_2^2$

$$2V = K q_1^2 + (K+A) q_2^2$$

$$= (q_1, q_2) (K \text{ } 0 \text{ } \text{ } 0 \text{ } K+A (q_1, q_2)$$

$$|V - \omega_2 n| = 0$$

$$2) \begin{vmatrix} K - m \omega_2^2 & 0 \\ 0 & (K+A) - \omega_2^2 \end{vmatrix} = 0$$

$$3) (K - m \omega_2^2) (K+A - m \omega_2^2) = 0$$

$$\therefore \omega_2 = \sqrt{\frac{K+A}{m}}$$

Comparing with earlier note $\omega_2 = \sqrt{\frac{K}{m}(1+\frac{2m}{M})}$

$$\therefore \frac{K+A}{m} = \frac{K}{m} + \frac{2m}{M}$$

$$2) \quad \frac{A}{m} = \frac{2m}{M} \therefore \boxed{A = \frac{2km}{M}}$$

$$T = \frac{1}{2} m \dot{\theta}_1^2 + \frac{1}{2} m \dot{\theta}_2^2$$

$$2T = (\dot{\theta}_1, \dot{\theta}_2) (0 \text{ } m \text{ } \text{ } 0 \text{ } m \text{ } \text{ } 0) (\dot{\theta}_1, \dot{\theta}_2)$$
\[ V = \frac{1}{2} mg l (\theta_1^2 + \theta_2^2) + \frac{1}{2} k l^2 (\theta_1^2 + \theta_2^2 - 2\theta_1 \theta_2) \]

\[ 2V = (mg + kl^2) (\theta_1^2 + \theta_2^2) + 2kl^2 \theta_1 \theta_2 \]

\[ = (\theta_1 \ \theta_2) \begin{pmatrix}
  mg + kl^2 & -kl^2 \\
  -kl^2 & mg + kl^2 \\
\end{pmatrix} (\theta_1 \ \theta_2) \]

\[ |V - W^2| = 0 \]

\[
\begin{vmatrix}
  mg + kl^2 - W^2ml^2 & -kl^2 \\
  -kl^2 & mg - kl^2 - W^2ml^2 \\
\end{vmatrix} = 0
\]

\[ (mg + kl^2 - W^2 ml^2)^2 - kl^4 y = 0 \]

\[ mg + kl^2 - W^2 ml^2 = \pm kl^2 \]

\[ W = \frac{mg + kl^2 + kl^2}{ml^2} \]

\[ \therefore W_1 = \sqrt{\frac{g}{l}}, \quad W_2 = \sqrt{\frac{g}{l + 2u/m}} \]

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Three Simple Pendulums Coupled to each other

i.e.

All three pendulums are connected to each other with springs having spring constant \( k \).

\[ T = \frac{1}{2} ml^2 (\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2) \]

\[ 2T = (\dot{\theta}_1 \ \theta_1 \ \dot{\theta}_3) \begin{pmatrix}
  ml^2 & 0 & 0 \\
  0 & ml^2 & 0 \\
  0 & 0 & ml^2 \\
\end{pmatrix} (\dot{\theta}_1 \ \theta_1 \ \dot{\theta}_3) \]
\[ V = \frac{1}{2} m g l (\theta_1 + \theta_2 + \theta_3) + \frac{1}{2} k l^2 \left[ (\theta_1 - \theta_2)^2 + (\theta_2 - \theta_3)^2 \right] \]

\[ 2V = \frac{1}{2} (m g l + k l^2) (\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2) - 2 k l^2 (\dot{\theta}_1 \dot{\theta}_2 + \dot{\theta}_2 \dot{\theta}_3 + \dot{\theta}_3 \dot{\theta}_1) \]

\[ = \begin{pmatrix} \theta_1 & \theta_2 & \theta_3 \end{pmatrix} \begin{pmatrix} m g l + k l^2 & -k l^2 & -k l^2 \\ -k l^2 & m g l + k l^2 & -k l^2 \\ -k l^2 & -k l^2 & m g l + k l^2 \end{pmatrix} \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{pmatrix} \]

\[ |V - W^2| = 0 \]

2) \[ \begin{vmatrix} m g l + k l^2 - m l w^2 & -k l^2 & -k l^2 \\ -k l^2 & m g l + k l^2 - m l w^2 & -k l^2 \\ -k l^2 & -k l^2 & m g l + k l^2 - m l w^2 \end{vmatrix} = 0 \]

3) \[ (m g l + k l^2 - m l w^2)^3 - 3 k l^2 (m g l + k l^2 - m l w^2) - 2 k l^2 = 0 \]

\[ x^3 - 3 k l^2 x - 2 k l^2 = 0 \]

\[ \text{where } x = (m g l + k l^2 - m l w^2) \]

4) \[ x^3 - w x^2 - 2 w x - 2 w^2 = 0 \]

5) \[ x (x+k) (x-k) - 2 k l (x+k) = 0 \]

6) \[ (x+k)^2 [x (x-k) + 2 k l] = 0 \]

7) \[ (x+k)^2 [x^2 - k x l - 2 k l x] = 0 \]
7) \((x + w) \left[ (n+w) - w\lambda - n\lambda^2 \right] \Rightarrow 0\)

8) \((x + w) \left[ (n+w) (x - k) - \lambda w (n+w) \right] \Rightarrow 0\)

9) \((x^2 + w) (n + k) (x - 2k) \Rightarrow 0\)

\[ x = -k \quad x = -k \quad x = 2kl \]

\[ mg + kl - mL = -kl \]

\[ w = \frac{mg + kl + kl}{ml^2} = \frac{mg + 2kl}{ml^2} \]

\[ w_1 = \sqrt{\frac{g}{k} + \frac{2k}{m}} \]

\[ w_2 = \sqrt{\frac{g}{k} + \frac{2k}{m}} \]

Similarly,

\[ w_3 = \sqrt{\frac{mg + kl - 2kl^2}{ml^2}} \]

\[ = \sqrt{\frac{g}{k} - \frac{k}{m}} \]