ASSIGNMENT XI

(Laurent Series, Singularity, Laurent series expansion, Concept of Residue, Classification of Residue, Calculation of Residue for quotient form)

1. The Laurent series expansion of $f(z) = \frac{2+3z}{z^2+z^3}$ about $z_0 = 0$ in the region $0 < |z| < 1$ is
   (a) $\boldsymbol{\frac{2}{z^2} + \frac{1}{z} - 1 + z - z^2 + \cdots}$ (b) $\frac{2}{z^2} - \frac{1}{z} + 1 - z + z^2 + \cdots$
   (c) $\frac{2}{z^2} + \frac{1}{z} + 1 - z + z^2 + \cdots$ (d) $\frac{2}{z^2} - \frac{1}{z} - 1 + z - z^2 + \cdots$

2. For $|z| < 1$ the series $z(1-z) + z^2(1-z) + z^3(1-z) + \cdots$
   (a) Converges (b) diverges (c) cannot be said

3. The Laurent series expansion of $f(z) = \frac{1}{z(z+5)}$ in the region $|z| < 5$ is
   (a) $\sum_{n=0}^{\infty} \frac{z^{n+1}}{5^{n-1}}$ (b) $\sum_{n=0}^{\infty} \frac{(-1)^n z^{n+1}}{5^{n-1}}$ (c) $\sum_{n=0}^{\infty} \frac{(-1)^n z^{n-1}}{5^{n+1}}$ (d) $\sum_{n=0}^{\infty} \frac{(-1)^{n+1} z^{n-1}}{5^{n+1}}$

4. The series expansion of the function $f(z) = \frac{1}{(z-1)(z-3)}$ in the region $0 < |z| < 1$ is
   (a) $\sum_{n=0}^{\infty} \frac{1}{2}\left(1 + \frac{1}{3^{n+1}}\right) z^n$ (b) $\sum_{n=0}^{\infty} \frac{1}{2}\left(1 - \frac{1}{3^{n+1}}\right) z^n$
   (c) $\sum_{n=0}^{\infty} \frac{1}{2}\left(1 + \frac{1}{3^{n+1}}\right) z^{n+1}$ (d) $\sum_{n=0}^{\infty} \frac{1}{2}\left(1 - \frac{1}{3^{n+1}}\right) z^{n+1}$

5. The series expansion of the function $f(z) = \frac{1}{(z-1)(z-3)}$ in the region $1 < |z| < 3$ is
   (a) $-\frac{1}{6}\sum_{n=0}^{\infty} \left(\frac{z}{3}\right)^n + \frac{1}{2z}\sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n$ (b) $-\frac{1}{6}\sum_{n=0}^{\infty} \left(\frac{z}{3}\right)^n - \frac{1}{2z}\sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n$
   (c) $\frac{1}{6}\sum_{n=0}^{\infty} \left(\frac{z}{3}\right)^n - \frac{1}{2z}\sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n$ (d) $\frac{1}{6}\sum_{n=0}^{\infty} \left(\frac{z}{3}\right)^n + \frac{1}{2z}\sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n$

6. The series expansion of the function $f(z) = \frac{1}{(z-1)(z-3)}$ in the region $|z| > 3$ is
   (a) $\frac{1}{2}\sum_{n=0}^{\infty}(3^n + 1)\frac{1}{z^{n+1}}$ (b) $\frac{1}{2}\sum_{n=0}^{\infty}(3^n + 1)\frac{1}{z^n}$
   (c) $\frac{1}{2}\sum_{n=0}^{\infty}(3^n - 1)\frac{1}{z^n}$ (d) $\frac{1}{2}\sum_{n=0}^{\infty}(3^n - 1)\frac{1}{z^{n+1}}$

7. The Laurent series representation of the function $f(z) = z^2 \sin\left(\frac{1}{z^2}\right)$ in the domain $0 < |z| < \infty$ is
   (a) $1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!}\frac{1}{z^{4n}}$ (b) $1 - \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!}\frac{1}{z^{4n}}$
   (c) $1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)!}\frac{1}{z^{4n}}$ (d) $1 - \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!}\frac{1}{z^{4(n-1)}}$

8. The representation of the function $f(z) = \frac{1}{1+z}$ in negative power of z that is valid when $1 < |z| < \infty$ is

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{z^n}$

(b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{z^n}$

(c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{z^{n+1}}$

(d) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{z^{n+1}}$

9. The Maclaurin series representation of the function $f(z) = \frac{z+1}{z-1}$ for $|z| < 1$ is

(a) $-1 - 2\sum_{n=1}^{\infty} z^n$

(b) $-1 + 2\sum_{n=1}^{\infty} z^n$

(c) $1 - 2\sum_{n=1}^{\infty} z^n$

(d) $1 + 2\sum_{n=1}^{\infty} z^n$

10. The Laurent series representation of the function $f(z) = \frac{z+1}{z-1}$ in the domain $1 < |z| < \infty$ is

(a) $-1 - 2\sum_{n=1}^{\infty} \frac{1}{z^n}$

(b) $-1 + 2\sum_{n=1}^{\infty} \frac{1}{z^n}$

(c) $1 + 2\sum_{n=1}^{\infty} \frac{1}{z^n}$

(d) $1 - 2\sum_{n=1}^{\infty} \frac{1}{z^n}$

Assignment 11

1. $$f(z) = \frac{2+3z}{z^2+z^3} \qquad 0 < |z| < 1$$

$$= \frac{1}{z^2}\left(\frac{2+3z}{1+z}\right) = \frac{1}{z^2}\left(3 - \frac{1}{1+z}\right)$$

$$= \frac{1}{z^2}\left(3 - \sum_{n=0}^{\infty} (-1)^n z^n\right)$$

$$= \frac{1}{z^2}\left(3 - 1 + z - z^2 + z^3 - \cdots\right)$$

$$= \frac{2}{z^2} + \frac{1}{z} - 1 + z - z^2 + \cdots \qquad \boxed{a}$$

2. The sum of the first n terms of the series

$$S_n = z - z^2 + z^2 - z^3 + z^3 - z^4 + \cdots + z^n - z^{n+1}$$

$$= z - z^{n+1}.$$

for $|z| < 1$, $z^{n+1} \to 0$ as $n \to \infty$

$$\therefore \lim_{n\to\infty} (S_n) = z.$$

i.e The given series converges for $|z| < 1$.

$\boxed{a}$

3. $$f(z) = \frac{1}{z(z+5)} \qquad |z| < 5.$$

$$= \frac{1}{z} \frac{1}{5(1+z/5)}$$

$$= \frac{1}{z}\cdot\frac{1}{5}\frac{1}{\{1-(-z/5)\}}$$

$$= \frac{1}{z}\cdot\frac{1}{5}\sum_{n=0}^{\infty}\left(-z/5\right)^n$$

$$\therefore f(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{n-1}}{5^{n+1}} \qquad \boxed{c}$$

4. $$f(z) = \frac{1}{(z-1)(z-3)} = \frac{1}{2}\left(\frac{1}{z-3} - \frac{1}{z-1}\right)$$

The function has singularities at $z=1$ and $z=3$.

for $0 < |z| < 1$,

$$f(z) = -\frac{1}{6}\left(1 - z/3\right)^{-1} + \frac{1}{2}(1-z)^{-1}$$

$$\therefore\ f(z) = -\frac{1}{6}\sum_{n=0}^{\infty}(z/3)^n + \frac{1}{2}\sum_{n=0}^{\infty} z^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{2}\left(1 - \frac{1}{3^{n+1}}\right) z^n \qquad \boxed{b}$$

5. for $1 < |z| < 3$.

$$f(z) = -\frac{1}{6}\left(1 - z/3\right)^{-1} - \frac{1}{2z}\left(1 - \frac{1}{z}\right)^{-1}$$

$$= -\frac{1}{6}\sum_{n=0}^{\infty}(z/3)^n - \frac{1}{2z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n \qquad \boxed{b}$$

6. $|z| > 3$,

$$f(z) = \frac{1}{2z}\left(1 - \frac{3}{z}\right)^{-1} - \frac{1}{2z}\left(1 - \frac{1}{z}\right)^{-1}$$

$$= \frac{1}{2z}\sum_{n=0}^{\infty}\left(\frac{3}{z}\right)^n - \frac{1}{2z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}(3^n - 1)\frac{1}{z^{n+1}} \qquad \boxed{d}$$

7.

We use the expansion

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

when $0 < |z| < \infty$

$$z^2 \sin\left(\frac{1}{z^2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\ \frac{1}{z^{4n}}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!}\ \frac{1}{z^{4n}} \qquad \boxed{a}$$

8.

$$\frac{1}{1+z} = \frac{1}{z}\,\frac{1}{1+\frac{1}{z}} = \frac{1}{z}\sum_{n=0}^{\infty}\left(-\frac{1}{z}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{z^{n+1}}$$

Replacing n by $n-1$ in this series and noting that $(-1)^{n-1} = (-1)^{n+1}$

then we have $$\frac{1}{1+z} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{z^n} \quad \boxed{b}$$

9.

$$f(z) = \frac{z+1}{z-1} \qquad |z| < 1$$

$$f(z) = \frac{z+1}{z-1} = -(z+1)\,\frac{1}{1-z} = (-z-1)\sum_{n=0}^{\infty} z^n$$

$$= -\sum_{n=0}^{\infty} z^{n+1} - \sum_{n=0}^{\infty} z^n$$

$$= -\sum_{n=1}^{\infty} z^n - \sum_{n=0}^{\infty} z^n$$

$$= -\sum_{n=1}^{\infty} z^n - \sum_{n=1}^{\infty} z^n - 1 = -1 - 2\sum_{n=1}^{\infty} z^n \quad \boxed{a}$$

10.

$$f(z) = \frac{z+1}{z-1} \qquad 1 < |z| < \infty.$$

$$f(z) = \frac{z+1}{z-1} = \frac{1+1/z}{1-1/z} = (1+1/z)\,\frac{1}{(1-1/z)} = (1+1/z)\sum_{n=0}^{\infty}(1/z)^n$$

$$= \sum_{n=0}^{\infty} 1/z^n + \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}$$

$$= \sum_{n=0}^{\infty}\frac{1}{z^n} + \sum_{n=1}^{\infty}\frac{1}{z^n} = 1 + 2\sum_{n=1}^{\infty}\frac{1}{z^n} \quad \boxed{c}$$