1. A bead of mass slides on a long straight wire which makes an angle $\alpha$ with, and rotates with constant angular velocity $\omega$ about the vertical axis as shown in the figure. Gravity $g$ acts vertically downwards

(a) Choose and appropriate generalized coordinate and find the Lagrangian.

(b) Write down explicit equations of motion.

Answer:

(a) Let the distance of the bead from the origin be $r$. Let us choose $r$ as the generalized coordinate. The spherical coordinates $(r, \theta, \phi)$ of the bead obey constraints

$$\theta = \alpha, \quad \phi = \omega t + \phi_0$$

The Lagrangian is given by

$$L = \frac{m}{2} \left( \dot{r}^2 + r^2 \omega^2 \sin^2 \alpha \right) - m g r \cos \alpha.$$ 

(b) There will be only one equation of motion, that is for $r$.

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{r}} \right) = \frac{d}{dt} (m \ddot{r}) = m \dddot{r}$$

and

$$\frac{\partial L}{\partial r} = m \omega^2 \sin^2 \alpha r - m g \cos \alpha$$

Thus,

$$m \dddot{r} = m \omega^2 \sin^2 \alpha r - m g \cos \alpha$$

2. Consider the motion of a free particle as viewed from a rotating coordinate system $(x', y', z')$ where

$$x' = x \cos \theta + y \sin \theta$$

$$y' = -x \sin \theta + y \cos \theta$$

$$z' = z$$

and $\theta = \theta(t)$ is some given function of time.

(a) Show that the new Lagrangian is given by

$$L = \frac{1}{2} m \left[ (\dot{x}'^2 + \dot{y}'^2 + \dot{z}'^2) + 2\omega (x' \dot{y}' - y' \dot{x}') + \omega^2 (x'^2 + y'^2) \right]$$

where $\omega = d\theta/dt$. 

(b) Write down the Lagrange’s equations and identify the pseudoforces, that is Coriolis force, centrifugal force and Euler force.

Answers:

(a) Inverse relations are

\[ \begin{align*}
  x &= x' \cos \theta - y' \sin \theta \\
  y &= x' \sin \theta + y' \cos \theta \\
  z &= z'
\end{align*} \]

Then, (substituting \( \omega = \dot{\theta} \))

\[ \begin{align*}
  \dot{x} &= x' \cos \theta - y' \sin \theta - (x' \sin \theta + y' \cos \theta) \omega \\
  \dot{y} &= x' \sin \theta + y' \cos \theta + (x' \cos \theta - y' \sin \theta) \omega
\end{align*} \]

Now, take

\[ \dot{x}^2 + \dot{y}^2 = \dot{x}'^2 + \dot{y}'^2 + \omega^2 (x'^2 + y'^2) + 2\omega (x'y' - y'x') \]

Now, write the new Lagrangian.

(b) Now,

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}'} \right) - \frac{\partial L}{\partial x'} = m\ddot{x}' - m\omega \dot{y}' - m\dot{\omega} y' - m\omega^2 x' = 0 \]

\[ \implies m\ddot{x}' = 2m\omega \dot{y}' + m\dot{\omega} y' + m\omega^2 x' \]

similarly

\[ m\ddot{y}' = -2m\omega \dot{y}' + m\dot{\omega} y' + m\omega^2 x' \]

we can identify terms: Coriolis force is proportional to \( \omega \), centrifugal force is identified by \( \omega^2 \) and the Euler force by \( \dot{\omega} \).

3. A heavy particle is placed at the top of a vertical hoop. Calculate the reaction of the hoop on the particle by means of the Lagrange’s undetermined multipliers and Lagrange’s equations. Find the height at which the particle falls off.

Answer:

We will use the polar coordinates \((r, \theta)\) for the position of the particle. Now, the constraint equation is \( r = \text{const} \), and in terms of infinitesimals

\[ dr + 0d\theta = 0 \]

That is \( a_r = 1 \) and \( a_\theta = 0 \). The Lagrangian is

\[ L = \frac{1}{2} m \left( \dot{r}^2 + r^2 \dot{\theta}^2 \right) - mgr \cos \theta \]

The equations of motion for \( r \) is

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{r}} \right) - \frac{\partial L}{\partial r} = \lambda_r a_r \]

\[ \implies m\ddot{r} - mr \ddot{\theta}^2 + mg \cos \theta = \lambda_r \\
\]

\[ \lambda_r = mg \cos \theta - mr \ddot{\theta}^2 \]
since \( \dot{r} = 0 \) and because energy is conserved (can also be obtained from \( \theta \) equation)

\[
\frac{m}{2} R^2 \dot{\theta}^2 + mgR \cos \theta = mgR \\
mR \dot{\theta}^2 = 2mg - 2mg \cos \theta
\]

Thus

\[
\lambda_r = (3 \cos \theta - 2) mg
\]

and \( \lambda_r \) is the normal reaction of the hoop in the outward direction and it cannot be negative. Hence,

\[
\theta_c = \cos^{-1}(2/3)
\]

4. A particle is moving on a straight line, in a force field given by the potential

\[
V(x) = V_0 \left(x^2 - a^2\right)^2
\]

where, \( V_0 \) and \( a \) are positive constants.

(a) Find the equilibrium positions and categorize those.

(b) Draw the phase space portrait.

Answers:

(a) Now, the equilibrium points are at \( x = 0 \) and \( x = \pm a \). The point at \( x = 0 \) is unstable equilibrium and the points \( x = \pm a \) are stable equilibria.

(b) Phase space portrait