1. [2 Marks] Two particles having Cartesian coordinates \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\), respectively, are attached to the extremities of a bar whose length \(l(t)\) changes with time in a prescribed fashion.

(a) Give the equations of constraint on the finite displacements of the Cartesian coordinates.

(b) Write the same constraint in terms of the infinitesimal displacements of these coordinates.

Answer:

(a) \((x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 = l(t)^2\)

(b) \((x_2 - x_1)(dx_2 - dx_1) + (y_2 - y_1)(dy_2 - dy_1) + (z_2 - z_1)(dz_2 - dz_1) - l\dot{l}dt = 0\)

2. [2 Marks] A dynamic system is subject to the constraint

\[\cos \theta dx + \sin \theta dy + (y \cos \theta - x \sin \theta) d\theta = 0.\]

To show that this constraint is holonomic, express this constraint as a function \(f(x, y, \theta) = 0\).

Answer:

\(f\) is found by integrating the constraint equation. Here mere inspection gives us,

\[f(x, y, \theta) = x \cos \theta + y \sin \theta\]

3. [3 Marks] The figure above shows an Atwood machine. Use \(x\) as the generalized coordinate.

(a) Write down virtual displacements of each particle in terms of \(\delta x\).

(b) Show all forces in free body diagram and mark those as constraint and applied forces.

(c) Using d’Alembert’s principle, find the acceleration of \(M_1\)

Answer:

If \(x_1\) and \(x_2\) are the coordinates of the two masses, then \(x_1 = x\) and \(x_2 = l - x\).

(a) \(\delta x_1 = \delta x\) and \(\delta x_2 = -\delta x\).

(b) The free body diagram is shown in the figure. \(T\) is the constraint force and \(M_1g\) and \(M_2g\) are the applied forces.

(c) According to d’Alembert’s principle

\[(M_1g - M_1\ddot{x}_1) \delta x_1 + (M_2g - M_2\ddot{x}_2) \delta x_2 = 0\]

\[(M_1g - M_1\ddot{x}) \delta x + (M_2g - M_2(\ddot{x})) (\delta x) = 0\]

\[(M_1 - M_2) g - (M_1 + M_2) \ddot{x} = 0\]

Thus, the acceleration of \(M_1\) is \((M_1 - M_2) g / (M_1 + M_2)\)
4. [6 Marks] Two masses \(m_1\) and \(m_2\) are connected by a inextensible string as shown in the figure (The wedge is fixed in space, \(m_1\) remains on the slope and \(m_2\) moves only in vertical direction, all surfaces are frictionless).

(a) How many degrees of freedom? Which generalized coordinate will you use?
(b) Write the Lagrangian of the system in terms of the generalized coordinate.
(c) Write the equation of motion.
(d) What is the generalized force on the system?

Answer:
(a) DoF=1, Generalized coordinate = \(s\)
(b) The transformation is given by

\[
\begin{align*}
  x_1 &= s \cos \alpha \\
  y_1 &= h - s \sin \alpha \\
  x_2 &= \text{constant} \\
  y_2 &= \text{const} + s
\end{align*}
\]

Then the lagrangian is

\[
L = \frac{1}{2} m_1 \left( \dot{x}_1^2 + \dot{y}_1^2 \right) + \frac{1}{2} m_2 \left( \dot{x}_2^2 + \dot{y}_2^2 \right) - m_1 g y_1 - m_2 g y_2 \\
= \frac{1}{2} \left( m_1 + m_2 \right) \dot{s}^2 + m_1 g s \sin \alpha - m_2 g s + \text{constant terms}
\]

(c) EoM

\[
\ddot{s} = \frac{m_1 \sin \alpha - m_2}{m_1 + m_2}
\]

(d) The gravity is already incorporated in Lagrangian. But, if we want to calculate the generalized force due to gravity, it will be given by

\[
Q = m_1 g (-\hat{j}) \cdot \frac{\partial \vec{r}_1}{\partial s} + m_2 g (-\hat{j}) \cdot \frac{\partial \vec{r}_2}{\partial s} \\
= m_1 g (-\hat{j}) \cdot \left( \cos \alpha \hat{i} - \sin \alpha \hat{j} \right) + m_2 g (-\hat{j}) \cdot \hat{j} \\
= (m_1 \sin \alpha - m_2) g
\]
5. [7 Marks] A system of two rigid rods (of equal length \( l \), mass \( m \) and uniform density), connected to each other end to end (see figure) by a frictionless rotating joint, is free to move in a 2D plane. The gravity is in the negative \( y \) direction.

(a) How many degrees of freedom for this system?
(b) Write the constraint equations.
(c) Suggest one set of generalized coordinates and then express cartesian coordinates coordinates of points \( A, B \) and \( C \) in terms of the generalized coordinates.
(d) Write the Lagrangian of the system.
(e) Write the equations of motion.
(f) What is the generalized force \( Q_j \) for each \( j \)?
(g) What is “geometry” of the accessible configuration space?

Answer:

(a) 4
(b) \( d(A, B) = l \) and \( d(B, C) = l \)
(c) Use coordinates \((x, y, \theta_A, \theta_C)\), as shown in the figure. Then,
\[
\begin{align*}
x_A &= x + l \cos \theta_A \\
y_A &= y + l \sin \theta_A \\
x_C &= x + l \cos \theta_C \\
y_C &= y + l \sin \theta_C
\end{align*}
\]
(d) Kinetic energy of each rod is
\[
\frac{1}{2} m \left( \frac{1}{4} (\dot{x} + \dot{x}_A)^2 + \frac{1}{4} (\dot{y} + \dot{y}_A)^2 \right) + \frac{1}{2} \frac{m l^2}{12} \dot{\theta}_A^2 \\
= \frac{1}{2} m \left( \frac{1}{4} (2\dot{x} - l \sin \theta_A \dot{\theta}_A)^2 + \frac{1}{4} (2\dot{y} + l \cos \theta_A \dot{\theta}_A)^2 \right) + \frac{1}{2} \frac{m l^2}{12} \dot{\theta}_A^2 \\
= \frac{1}{2} m \left( \dot{x}^2 + \dot{y}^2 + \frac{1}{4} l^2 \dot{\theta}_A^2 + (\dot{y} \cos \theta_A - \dot{x} \sin \theta_A) l \dot{\theta}_A \right) + \frac{1}{2} \frac{m l^2}{12} \dot{\theta}_A^2
\]
The Lagrangian is
\[
L = m (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} \frac{m l^2}{3} (\dot{\theta}_A^2 + \dot{\theta}_B^2) + \frac{m}{2} (\dot{y} \cos \theta_A - \dot{x} \sin \theta_A) l \dot{\theta}_A + \frac{m}{2} (\dot{y} \cos \theta_B - \dot{x} \sin \theta_B) l \dot{\theta}_B \\
- mg \left( y + \frac{l}{2} (\sin \theta_A + \sin \theta_C) \right)
\]
(e) Equations of motion:

\[
\begin{align*}
\frac{d}{dt} \left( m\dot{x} - ml \left( \sin \theta_A \dot{\theta}_A + \sin \theta_B \dot{\theta}_B \right) \right) &= 0 \\
\frac{d}{dt} \left( mg + ml \left( \cos \theta_A \dot{\theta}_A + \cos \theta_B \dot{\theta}_B \right) \right) &= -mg \\
\frac{d}{dt} \left( \frac{ml^2}{3} \dot{\theta}_A + \frac{1}{2} (\dot{y} \cos \theta_A - \dot{x} \sin \theta_A) l \right) &= \frac{m}{2} (-\dot{y} \sin \theta_A - \dot{x} \cos \theta_A) l \dot{\theta}_A - \frac{mgl}{2} \cos \theta_A \\
\frac{d}{dt} \left( \frac{ml^2}{3} \dot{\theta}_B + \frac{1}{2} (\dot{y} \cos \theta_B - \dot{x} \sin \theta_B) l \right) &= \frac{m}{2} (-\dot{y} \sin \theta_B - \dot{x} \cos \theta_B) l \dot{\theta}_A - \frac{mgl}{2} \cos \theta_B
\end{align*}
\]

(f) Generalized force is

\[
\begin{align*}
Q_x &= 0 \\
Q_y &= 2mg \\
Q_{\theta_A} &= mgl \cos \theta_A \\
Q_{\theta_B} &= mgl \cos \theta_B
\end{align*}
\]

(g) The geometry is a direct product of a plane and a 2-torus.