Solution of Assignment 7

7.1 For a uniform cantilever Euler-Bernoulli beam of linear density $\rho A$, length $l$ and flexural stiffness $EI$ carrying a point mass $m$ at the free end (at $x = l$), the transverse displacement is assumed as $w(x, t)$. The equation of motion for the transverse vibration of the beam is given by

$$\rho Aw_{tt} + EIw_{xxxx} = 0$$

and the boundary conditions are $w(0, t) = 0$, $w_x(0, t) = 0$, $EIw_{xx}(l, t) = 0$, and $EIw_{xxx}(l, t) = mw_{tt}$.

Let, the solution for the field variable as $w(x, t) = W(x)e^{i\omega t}$. Thus the Eigen value problem is formulated as

$$-\omega^2 W\rho A + EIW'''' = 0$$

and the boundary conditions are $W(0) = 0$, $W'(0) = 0$, $W''(l) = 0$, $EIW'''(l) = -m\omega^2 W(l)$.

Now, the solution for $W(x)$ is assumed as $W(x) = B_1 \cosh \beta x + B_2 \sinh \beta x + B_3 \cos \beta x + B_4 \sin \beta x$, where $\beta^4 = \frac{\rho A\omega^2}{EI}$.

Substituting the solution form in the boundary conditions, we get

$$\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
\cosh \beta l & \sinh \beta l & -\cos \beta l & -\sin \beta l \\
\sinh \beta l + \frac{m\beta}{\rho A} \cosh \beta l & \cosh \beta l + \frac{m\beta}{\rho A} \sinh \beta l & \sin \beta l + \frac{m\beta}{\rho A} \cos \beta l & -\cos \beta l + \frac{m\beta}{\rho A} \sin \beta l
\end{bmatrix}
\begin{bmatrix}
B_1 \\
B_2 \\
B_3 \\
B_4
\end{bmatrix}
= \begin{bmatrix}0 \\
0 \\
0 \\
0 \end{bmatrix}$$

For non-trivial solution to exist, we must have the determinant of the above 4 by 4 matrix to be zero. This leads to the characteristic equation, which is

$$1 + \cos \beta l \cosh \beta l + \frac{m\beta}{\rho A} (\cos \beta l \sinh \beta l - \sin \beta l \cosh \beta l) = 0$$

7.2 For a uniform fixed-pinned Euler-Bernoulli beam of linear density $\rho A$, length $l$ and flexural stiffness $EI$, the transverse displacement is assumed as $w(x, t)$. The equation of motion for the transverse vibration of the beam is given by

$$\rho Aw_{tt} + EIw_{xxxx} = 0$$

and the boundary conditions are $w(0, t) = 0$, $w_x(0, t) = 0$, $w(l, t) = 0$, and $EIw_{xx}(l, t) = 0$.

Let, the solution for the field variable as $w(x, t) = W(x)e^{i\omega t}$. Thus the Eigen value problem is formulated as

$$-\omega^2 W\rho A + EIW'''' = 0$$

and the boundary conditions are $W(0) = 0$, $W'(0) = 0$, $W''(l) = 0$, $EIW'''(l) = -m\omega^2 W(l)$.

Now, the solution for $W(x)$ is assumed as $W(x) = B_1 \cosh \beta x + B_2 \sinh \beta x + B_3 \cos \beta x + B_4 \sin \beta x$, where $\beta^4 = \frac{\rho A\omega^2}{EI}$.

Substituting the solution form in the boundary conditions, we get

$$\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
\cosh \beta l & \sinh \beta l & \cos \beta l & \sin \beta l \\
\cosh \beta l & \sinh \beta l & -\cos \beta l & -\sin \beta l
\end{bmatrix}
\begin{bmatrix}
B_1 \\
B_2 \\
B_3 \\
B_4
\end{bmatrix}
= \begin{bmatrix}0 \\
0 \\
0 \\
0 \end{bmatrix}$$
For non-trivial solution to exist, we must have the determinant of the above 4 by 4 matrix to be zero. This leads to the characteristic equation, which is
\[ \tan \beta l \tanh \beta l = 0. \]

The first solution of the characteristic equation is \( \beta_1 l = 3.9266 \). Hence, the first fundamental frequency is given as
\[ \omega_1 = \beta_1^2 \sqrt{\frac{EI}{\rho A}} = 15.42 \sqrt{\frac{EI}{\rho A}}. \]

7.3 For discretizing the system, the transverse displacement of the uniform simply-supported Euler-Bernoulli beam having linear density \( \rho A \), length \( l \) and flexural stiffness \( EI \), is assumed as \( w(x, t) = \sum_{i=1}^{2} a_i(t) \phi_i(x) = a_1(t)x(l - x) + a_2(t)x^2(l - x). \)

The Lagrangian of the system is given as
\[ L = T - V = \frac{1}{2} \int_0^l (\rho A w^2_t - EI w^2_{xx}) dx = \frac{1}{2} \int_0^l (\rho A(a_1 \phi_1 + a_2 \phi_2)^2 - EI(a_1 \phi_1'' + a_2 \phi_2'')^2) dx. \]

Now, assuming the solutions of the form \( a_1 = A_1 e^{i\omega t} \) and \( a_2 = A_2 e^{i\omega t} \), for non-trivial solution of \( A_1 \) and \( A_2 \), the following determinant must vanish
\[ \begin{vmatrix} EIl & \left[ \frac{4}{2l} \frac{2l}{4l^2} \right] \\ \left[ \frac{1}{30} \frac{1}{60} \frac{1}{105} \right] & \rho A^5 \omega^2 \end{vmatrix} = 0. \]

The solutions of this characteristic equation give the first two fundamental natural frequencies of the discretized system. They are obtained as
\[ \omega_1 = 10.96 \frac{\sqrt{EI}}{l^2 \sqrt{\rho A}} \]
\[ \omega_2 = 50.2 \frac{\sqrt{EI}}{l^2 \sqrt{\rho A}} \]

7.4 For a simply-supported uniform beam, the equation of motion for the transverse vibration is given by
\[ \rho A w_{tt} + EI w_{xxxx} = 0. \]

The boundary conditions are \( w(0, t) = 0, w(l, t) = 0, w_{xx}(0, t) = 0, \) and \( w_{xx}(l, t) = 0. \)

The beam was initially loaded by a constant distributed force \( q(x, t) = Q_0 \) and the load is suddenly removed at time \( t = 0. \) The initial deflected shape of the beam at time \( t = 0 \) is given by \( w(x, 0) = \frac{Q_0}{6EI}(x^4 - 2x^3l + lx^2) \) and the initial velocity is given by \( w_t(x, 0) = 0. \)

Let, the solution form for the field variable be
\[ w(x, t) = \sum_{j=1}^{\infty} (C_j \cos \omega_j t + D_j \sin \omega_j t) W_j(x). \]
where, the Eigen function for the simple-supported Euler-Bernoulli beam is $W_j(x) = \sin \frac{j\pi x}{l}$. Hence, $w(x, t) = C_j \sin \frac{j\pi x}{l} \cos \omega_j t$. Now using the initial condition $w_t(x, 0) = 0$, we get $D_j = 0$. Using the initial condition $w(x, 0)$, we get

$$w(x, 0) = \sum_{j=1}^{\infty} C_j \sin \frac{j\pi x}{l} = \frac{Q_0}{24EI} (x^4 - 2x^3 l + xl^3).$$

Now, for solving $C_j$, we follow the standard procedure and multiply both sides by $\sin \frac{i\pi x}{l}$ and integrate over the domain of the problem. Using orthogonality principle, we get

$$C_i = -\frac{2Q_0 l^4}{EI\pi^5} \frac{(\cos i\pi - 1)}{i^5}.$$

Hence, the solution for the field variable becomes

$$w(x, t) = -\frac{2Q_0 l^4}{EI\pi^5} \sum_{j=1}^{\infty} \frac{\cos j\pi - 1}{j^5} \sin \frac{j\pi x}{l} \cos \omega_j t = \frac{4Q_0 l^4}{EI\pi^5} \sum_{j=1,3,...}^{\infty} \frac{1}{j^5} \sin \frac{j\pi x}{l} \cos \omega_j t$$

($\cos j\pi = 1$ for even values of $j$, and $\cos j\pi = -1$ for odd values of $j$)