Solution of Assignment 5

1. Ans:b DEOM for transverse vibrations of string can be written as,

\[ \rho Aw_{tt} - Tw_{xx} = \mathcal{R}[Q(x)e^{i\Omega t}] \]  \hfill (1)

Boundary value problem of (1) can be formulated as,

\[ -\Omega^2 \mu(x) X(x) + \mathcal{K}[X(x)] = Q(x) \]  \hfill (2)

where, \( \mu(x) \) is the unknown amplitude function of particular solution of (1). Let \( G(x, \bar{x}, \Omega) \) be the solution of (2) with concentrated unit force at \( x = \bar{x} \in [0,l] \). So, (2) can be modified as,

\[ -\Omega^2 G(x, \bar{x}, \Omega) - c^2 G_{xx}(x, \bar{x}, \Omega) = \frac{1}{\rho A} \delta(x - \bar{x}) \]  \hfill (3)

where, \( \mu(x) = 1 \) and \( \mathcal{K}[w] = -c^2 w_{xx} \). We can consider two regions of string as follows:

\[ -\Omega^2 G - c^2 G_{xx} = 0, \quad 0 \leq x < \bar{x} \]
\[ -\Omega^2 G - c^2 G_{xx} = 0, \quad \bar{x} \leq x < l \]  \hfill (4)

Solution of (4) can be written as,

\[ G(x, \bar{x}, \Omega) = A_L \sin \frac{\Omega x}{c} + B_L \cos \frac{\Omega x}{c}, \quad 0 \leq x < \bar{x} \]
\[ = A_R \sin \frac{\Omega x}{c} + B_R \cos \frac{\Omega x}{c}, \quad \bar{x} \leq x < l \]  \hfill (5)

Boundary conditions to be used are as follows:

\[ G_x(0, \bar{x}, \Omega) = 0 \]
\[ G(l, \bar{x}, \Omega) = 0 \]  \hfill (6)

Along with (6), continuity and force balance at \( x = \bar{x} \) will be used.

\[ G_x(\bar{x}^-, \bar{x}, \Omega) = G_x(\bar{x}^+, \bar{x}, \Omega) \]
\[ \int_0^l (-\Omega^2 G - c^2 G_{xx}) dx = \frac{1}{\rho A} \int_0^l \delta(x - \bar{x}) dx = \frac{1}{\rho A} \]  \hfill (7)

Integrand at LHS of force balance equation is zero except at \( x = \bar{x} \), so we can write,

\[ \lim_{\epsilon \to 0} \int_{\bar{x} - \epsilon}^{\bar{x} + \epsilon} (\Omega^2 G + c^2 G_{xx}) dx = -\frac{1}{\rho A} \]  \hfill (8)
As $G$ is continuous function of $x$, so its integration over such a small interval $(2\epsilon)$ will be zero. So $\Omega^2 G$ will not contribute anything in the integral. so (8) becomes,

$$\lim_{\epsilon \to 0} c^2 \int_{\bar{x}-\epsilon}^{\bar{x}+\epsilon} (G_{xx}) dx = -\frac{1}{\rho A}$$

$$\Rightarrow \lim_{\epsilon \to 0} c^2 [G_{x}(\bar{x} + \epsilon) - G_{x}(\bar{x} - \epsilon)] = -\frac{1}{\rho A}$$

$$\Rightarrow c^2 [G_{x}(\bar{x}^+, \bar{x}, \Omega) - G_{x}(\bar{x}^-, \bar{x}, \Omega)] = -\frac{1}{\rho A} \quad (9)$$

(5) and (6) gives,

$$A_L = 0, \quad A_R \sin \frac{\Omega l}{c} + B_R \cos \frac{\Omega l}{c} = 0 \quad (10)$$

Substituting (5) in continuity equation at $x = \bar{x}$ (7), we get,

$$(B_R - B_L) \cos \frac{\Omega \bar{x}}{c} + A_R \sin \frac{\Omega \bar{x}}{c} = 0 \quad (11)$$

Substituting (5) in force balance equation (9), we get,

$$-(B_R - B_L) \sin \frac{\Omega \bar{x}}{c} + A_R \cos \frac{\Omega \bar{x}}{c} = -\frac{1}{\rho A \Omega c} \quad (12)$$

Solving (10), (11) and (12), we get,

$$A_R = -\frac{1}{\rho A \Omega c} \cos \frac{\Omega \bar{x}}{c}$$

$$B_R = \frac{1}{\rho A \Omega c} \cos \frac{\Omega \bar{x}}{c} \tan \frac{\Omega l}{c}$$

$$B_L = \frac{1}{\rho A \Omega c} \sin \frac{\Omega (l - \bar{x})}{c} \sec \frac{\Omega l}{c} \quad (13)$$

Using (10) and (13), we can write,

$$G(x, \bar{x}, \Omega) = \begin{cases} 
\sin \frac{\Omega (l - \bar{x})}{c} \cos \frac{\Omega x}{c}, & \text{for } 0 \leq x \leq \bar{x} \\
\rho A \Omega c \cos \frac{\Omega l}{c} \sin \frac{\Omega (l - x)}{c} \cos \frac{\Omega \bar{x}}{c}, & \text{for } \bar{x} \leq x \leq l 
\end{cases}$$
2. Ans: \textbf{d} Statement A and C both are correct. Refer lecture 15 for the details.

3. Ans: \textbf{a} Boundary conditions for this problem are,

\begin{align*}
EAu_x(0, t) &= d_1 u_x(0, t) \\
EAu_x(l, t) &= -d_2 u_x(l, t)
\end{align*}

(14)

Let displacement field \( u(x, t) \) for longitudinal vibrations of bar is \( u(x, t) = U(x) e^{st} \) where \( s \) is the complex number. Substituting the displacement form in DEOM of free vibration, we get ordinary differential equation in \( U(x) \). Solution of that can be written as,

\[ U(x) = Be^{sx/c} + Ce^{-sx/c} \]  

(15)

From (14) and (15), we get algebraic equations in \( B \) and \( C \).

\begin{align*}
B(1 - a_1) - C(1 + a_1) &= 0 \\
B(1 + a_2)e^\gamma - C(1 - a_2)e^{-\gamma} &= 0
\end{align*}

(16)

where, \( \gamma = \frac{sl}{c} \) and \( a_i = \frac{cd_i}{EA} \). For non-trivial solutions of (16), determinant of coefficient matrix should vanish. It gives,

\[ e^{2\gamma} = \frac{(a_1 - 1)(a_2 - 1)}{(a_1 + 1)(a_2 + 1)} \]

4. Ans: \textbf{c} Substitution of assumed general solution form \( w(x, t) = a(t) \sin \frac{\pi x}{l} \) in DEOM \( \rho A w_{tt} - Tw_{xx} + dw_{tt} = 0 \), we get,

\[ \ddot{a} + \frac{d}{\rho A} \dot{a} + \frac{\pi^2 c^2}{l^2} a = 0 \]

where \( c^2 = \frac{T}{\rho A} \). Comparing it with standard form of DEOM for free damped vibration \( \ddot{a} + 2\xi \omega \dot{a} + \omega^2 a = 0 \), we get,

\[ \omega = \frac{\pi c}{l} \text{ and } \xi = \frac{dl}{2\pi \rho Ac} \]