Solution of Assignment 2

1. Ans: (A) Using Hamilton’s principle for conservative system,

\[ \delta \int_{t_1}^{t_2} (K - V) dt = 0 \]  

(1)

where, \( \delta K = \int_0^l [\rho A + m\delta(x - l/2)]w_{,t} \delta w_{,t} dx \) and \( \delta V = \int_0^l T w_{,x} \delta w_{,x} dx \)

So, \( \delta V \) can be integrated by parts which gives,

\[ \delta V = T w_{,x} \delta w \bigg|_0^l - \int_0^l T w_{,xx} \delta w dx \]

Now,

\[ \int_{t_1}^{t_2} \delta K dt = \int_{t_1}^{t_2} \int_0^l [\rho A + m\delta(x - l/2)]w_{,t} \delta w_{,t} dt dx \]

which upon integrating by parts and taking variation of dependent variable (\( \delta w \)) at time instants \( t_1 \) and \( t_2 \) as zero, becomes

\[ \int_{t_1}^{t_2} \delta K dt = - \int_{t_1}^{t_2} \int_0^l [\rho A + m\delta(x - l/2)]w_{,tt} \delta w dt dx \]

Similarly for potential term,

\[ \int_{t_1}^{t_2} \delta V dt = \int_{t_1}^{t_2} \left[ T w_{,x} \delta w \bigg|_0^l - \int_0^l T w_{,xx} \delta w dx \right] dt \]

Putting kinetic and potential terms in equation (1) and rearranging,

\[ \int_{t_1}^{t_2} \left\{ - T w_{,x} \delta w \bigg|_0^l + \int_0^l \left[ T w_{,xx} - (\rho A + m\delta(x - l/2))w_{,tt} \right] \delta w dx \right\} dt = 0 \]  

(2)

Since, \( \delta w \) is arbitrary over the interval \( 0 < x < l \) and for any chosen time instants, equation (2) can be satisfied only when its individual terms are equal to zero. It gives equation of motion and general boundary conditions.

\[ T w_{,xx} = (\rho A + m\delta(x - l/2))w_{,tt} \]  

(3)
\[ Tw_x \delta w = 0, \quad x = 0 \]
\[ Tw_x \delta w = 0, \quad x = l \]  \hspace{1cm} (4)

2. Ans: (B) Equation of motion (EOM) for taut string undergoing transverse vibrations,

\[ c^2 w_{xx} = w_{tt} \]  \hspace{1cm} (5)

where, \( c = \sqrt{T/\rho} \). Solving the differential EOM using separation of variables, let

\[ w(x, t) = W(x)S(t) \]

Substituting this into EOM and rearranging it gives,

\[ \ddot{S} S = c^2 W'' W = -\omega^2 \text{ (Let)} \]

Solution of both 2nd-order ODE can be written as,

\[ W(x) = A \cos \left( \frac{\omega}{c} x \right) + B \sin \left( \frac{\omega}{c} x \right) \]
\[ S(t) = C \cos \omega t + D \sin \omega t \]  \hspace{1cm} (6)

Boundary conditions: \( Tw_x(0) = k_1 w(0) \) and \( Tw_x(l) = -k_2 w(l) \)

BCs can also be written as,

\[ TW'(0) = k_1 W(0) \]
\[ TW'(l) = -k_2 W(l) \]  \hspace{1cm} (7)

Putting value of \( W(x) \) from (6) into BCs (7),

\[ k_1 A - \frac{T\omega}{c} B = 0 \]
\[ \left[ -\frac{T\omega}{c} \sin \frac{\omega l}{c} + k_2 \cos \frac{\omega l}{c} \right] A + \left[ \frac{T\omega}{c} \cos \frac{\omega l}{c} + k_2 \sin \frac{\omega l}{c} \right] B = 0 \]

For non-trivial solution, determinant of coefficients of \( A \) and \( B \) should vanish. It gives,

\[ \left| \begin{array}{cc}
-\frac{T\omega}{c} \sin \frac{\omega l}{c} + k_2 \cos \frac{\omega l}{c} & \frac{T\omega}{c} \cos \frac{\omega l}{c} + k_2 \sin \frac{\omega l}{c} \\
\frac{T\omega}{c} \sin \frac{\omega l}{c} + k_2 \cos \frac{\omega l}{c} & -\frac{T\omega}{c} \sin \frac{\omega l}{c} + k_2 \cos \frac{\omega l}{c}
\end{array} \right| = 0 \]  \hspace{1cm} (8)
After solving and rearranging it, we get the characteristic or frequency equation.

\[ \tan \frac{\omega l}{c} = \frac{(k_1 + k_2) T \omega}{c \left( \frac{T \omega}{c} \right)^2 - k_1 k_2} \]

3. Ans: (d) Similar to previous problem solution, using BCs,

\[ mW(0) \ddot{S}(t) = TW'(0)S(t) \]
\[ TW'(l) = -kW(l) \]

Using \( \ddot{S}(t) = -\omega^2 \) and solution of \( W(x) \) we get,

\[ m\omega c A + TB = 0 \]
\[ \left[ -\frac{T \omega}{c} \sin \frac{\omega l}{c} + k \cos \frac{\omega l}{c} \right] A + \left[ \frac{T \omega}{c} \cos \frac{\omega l}{c} + k \sin \frac{\omega l}{c} \right] B = 0 \]

For non-trivial solution of \( A \) and \( B \), determinant of its coefficients should vanish. So we get,

\[ \left| \begin{array}{ccc} m\omega c & -\frac{T \omega}{c} \sin \frac{\omega l}{c} + k \cos \frac{\omega l}{c} & \frac{T \omega}{c} \cos \frac{\omega l}{c} + k \sin \frac{\omega l}{c} \\ -\frac{T \omega}{c} \sin \frac{\omega l}{c} + k \cos \frac{\omega l}{c} & m\omega c & T \frac{T \omega}{c} \cos \frac{\omega l}{c} + k \sin \frac{\omega l}{c} \end{array} \right| = 0 \]

After solving and rearranging it, we get the characteristic equation.

\[ \tan \frac{\omega l}{c} = \frac{T(k - m\omega^2)}{\omega(mkc + T^2/c)} \]

4. Ans: (c) Equation of motion (EOM) for uniform and homogeneous bar undergoing longitudinal vibrations,

\[ c^2 u_{xx} = u_{tt} \]

where, \( c = \sqrt{E/\rho} \) for \( \rho \) be the volume density.

Similar to previous solutions, here also field variable \( u(x, t) \) can be separated into spatial \( U(x) \) and temporal \( T(t) \) variables. Boundary conditions for given system are,

\[ u(0, t) = 0 \]
\[ EA u_x(l, t) = -M u_{tt}(l, t) - ku(l, t) \]
which can be modified as,

\[ U(0) = 0 \]

\[ EAU'(l)T(t) = -MU(l)\ddot{T}(t) - kU(l)T(t) \]  

(12)

Substituting \( \ddot{T}(t) = -\omega^2 \) and solution form of \( W(x) \) we get,

\[ 1.A + 0.B = 0 \]

\[ \begin{bmatrix}
-\frac{EA\omega}{c} \sin \frac{\omega l}{c} + (k - M\omega^2) \cos \frac{\omega l}{c} & \frac{EA\omega}{c} \cos \frac{\omega l}{c} + (k - M\omega^2) \sin \frac{\omega l}{c} \\
-\frac{EA\omega}{c} \sin \frac{\omega l}{c} + (k - M\omega^2) \cos \frac{\omega l}{c} & \frac{EA\omega}{c} \cos \frac{\omega l}{c} + (k - M\omega^2) \sin \frac{\omega l}{c}
\end{bmatrix} A + B = 0 \]

(13)

For non-trivial solution of \( A \) and \( B \), determinant of its coefficients should vanish. So we get,

\[ \begin{vmatrix}
1 & 0 \\
-\frac{EA\omega}{c} \sin \frac{\omega l}{c} + (k - M\omega^2) \cos \frac{\omega l}{c} & \frac{EA\omega}{c} \cos \frac{\omega l}{c} + (k - M\omega^2) \sin \frac{\omega l}{c}
\end{vmatrix} = 0 \]

After solving and rearranging it, we get the characteristic equation.

\[ \tan \alpha = \frac{EAl\alpha}{M\alpha^2c^2 - kl^2} \]

where \( \alpha = \frac{\omega l}{c} \). Here \( \omega \) is nth natural frequency.

5. Ans: (B) Characteristic equation for fixed - free bar is, \( \cos \alpha = 0 \) while for fixed - attached mass (\( M \)) bar is, \( \alpha \tan \alpha = \beta \) where \( \alpha = \frac{\omega l}{c} \) and \( \beta = \frac{m}{M} \). As per given condition, for given length (\( l \)) and material properties (\( c = \sqrt{E/\rho} \))

\[ \alpha_1(\text{new}) = \frac{1}{2}\alpha_1(\text{old}) \]

\[ \beta = \pi/4 \]

For fixed - free bar, \( \alpha_1(\text{old}) = \pi/2 \) so for fixed - attached mass bar \( \alpha_1(\text{new}) = \pi/4 \). Using frequency equation of bar with mass,

so, \( M = 4m/\pi = 1.273m \)