1. (a) Compute the bases of $\text{Ker } f$ and $\text{Im } f$ where $f : \mathbb{R}^3 \to \mathbb{R}^3$ with $f(x_1, x_2, x_3) = (x_1 + 2x_2 + 3x_3, x_1 + 3x_2 + 2x_3, x_1 + x_2)$. (2)

(b) Let $f : V \to W$ be a homomorphism of finite dimensional vector spaces.
   
   i. Show that $f$ is injective if and only if there exists a homomorphism $g : W \to V$ such that $g \circ f = id_V$. (2)
   
   ii. Show that $f$ is surjective if and only if there exists a homomorphism $h : W \to V$ such that $f \circ h = id_W$. (2)

(c) Let $h : D \to D'$ be an arbitrary map. For every field $K$, the map $h^* : K^D \to K^{D'}$ defined by $g \mapsto g \circ h$ is $K$-linear. Describe the functions in $\text{Ker } h^*$ and in $\text{Im } h^*$. Show that $h^*$ is injective if and only if $h$ is surjective. (4)

(d) For every $K$-vector space $V$, the map $f \mapsto f(1)$ is a $K$-isomorphism of $\text{Hom}_K(K, V)$ onto $V$. (4)

2. Let $I \subseteq \mathbb{R}$ be an interval with more than one point and let $a \in I$. For $n \in \mathbb{N}^*$, let

$$T_{a,n} : C^{n-1}_K(I) \to K[t]_n$$

be the map which maps every function $f \in C^{n-1}_K(I)$ to its Taylor-polynomial of degree $< n$ of $f$ at $a$, i.e.,

$$f \mapsto T_{a,n}(f) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (t-a)^k.$$

Show that $T_{a,n}$ is $K$-linear$^2$. Determine the kernel and prove that the map $T_{a,n}$ is surjective. (Hint: For every sequence $a_n, n \in \mathbb{N}$ of real or

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$^1$ $C^{n-1}_K$ is set of all $n-1$ times differentiable functions

$^2$ $f^{(k)}$ means $f$ is differentiated $k$ times
complex numbers, there exists an infinitely many times differentiable function $f$ on $\mathbb{R}$ with values in $\mathbb{R}$ (resp. $\mathbb{C}$) such that for all $n \in \mathbb{N}, f^n(0) = a_n$ \hfill (6)
1. Let $\mathbb{R}$ denote the field of real numbers. Determine whether the following maps are $\mathbb{R}$-linear:

   (a) $f : \mathbb{R}^2 \to \mathbb{R}^2$ with $f(x_1, x_2) := (x_1^2, x_2)$.
   (b) $f : \mathbb{R}^2 \to \mathbb{R}^2$ with $f(x_1, x_2) := (x_1 + 1, 0)$.
   (c) $f : \mathbb{R}^2 \to \mathbb{R}^2$ with $f(x_1, x_2) := (x_1 + x_2, x_1)$.
   (d) $f : \mathbb{R}^3 \to \mathbb{R}^2$ with $f(x_1, x_2, x_3) := (|x_1 - x_2|, x_3)$.
   (e) $f : \mathbb{R}^3 \to \mathbb{R}^2$ with $f(x_1, x_2, x_3) := (3x_1 + 2x_2, x_1 + x_3)$.

2. Give examples of two endomorphisms $f$ and $g$ of an infinite dimensional vector space such that

   (a) $f$ is injective, but not surjective.
   (b) $g$ is surjective, but not injective.

3. Let $V$ be a finite dimensional $K$-vector space and let $U, W$ be subspaces of $V$ of equal dimension. Then there exists a $K$-automorphism $f$ of $V$ such that $f(U) = W$.

4. Let $f : V \to W$ be a homomorphism of finite dimensional $K$-vector spaces. Show that

   (a) $f$ is injective if and only if there exists a homomorphism $g: W \to V$ such that $g \circ f = \text{id}_V$.
   (b) $f$ is surjective if and only if there exists a homomorphism $h: W \to V$ such that $f \circ h = \text{id}_W$.

5. Let $V$ be a finite dimensional $K$-vector space and let $f : V \to V$ be an endomorphism of $V$. Show that the following statements are equivalent

   (a) $f$ is not an automorphism of $V$.
   (b) There exists an $K$-endomorphism $g \neq 0$ of $V$ such that $g \circ f = 0$.
   (c) There exists an $K$-endomorphism $h \neq 0$ of $V$ such that $f \circ h = 0$. 
