Linear Algebra: Exercise 3 - to be submitted
Maximum marks: 20

1. Let $x_1, \ldots, x_n$ be a basis of the $K$-vector space $V$ and let $a_{ij} \in K$, $1 \leq i \leq j \leq n$. Show that

$$y_1 = a_{11}x_1, \ y_2 = a_{12}x_1 + a_{22}x_2, \ldots, \ y_n = a_{1n}x_1 + a_{2n}x_2 + \cdots + a_{nn}x_n$$

is a basis of $V$ if and only if $a_{11} \cdots a_{nn} \neq 0$. – (10 marks).

2. Let $K$ be a field and let $K[X]$ (respectively, $K[X]_m$, $m \in \mathbb{N}$) be the $K$-vector space of all polynomials (respectively, polynomials of degree $< m$) with coefficients in $K$. Let $f_n \in K[X]$, $n \in \mathbb{N}$, be a sequence of polynomials with $\deg f_n \leq n$ for all $n \in \mathbb{N}$. Show that for every $m \in \mathbb{N}$, $f_0, \ldots, f_{m-1}$ is a $K$-basis of the subspace $K[X]_m$ if and only if $\deg f_n = n$ for all $n = 0, \ldots, m - 1$. (Hint: Use question 1. – (10 marks)
1. Let \( V \) be a vector space over a field \( K \).
   
   (a) If \( V \) has a finite generating system, then every generating system of \( V \) has a finite generating system.
   
   (b) If \( v_i, i \in I \), is a generating system for \( V \), then every maximal linearly independent subsystem of \( v_i, i \in I \), is a basis of \( V \).

2. Determine which of the following systems of functions are linearly independent over \( \mathbb{R} \) in the \( \mathbb{R} \)-vector space \( \mathbb{R}^\mathbb{R} \) of all functions:
   
   (a) 1, \( \sin t \), \( \cos t \).
   
   (b) \( \sin t \), \( \cos t \), \( \sin(\alpha + t) \) (\( \alpha \in \mathbb{R} \) fixed).
   
   (c) \( t \), \( |t| \), \( \text{Sign} t \). (\( \text{Sign}(x) = 1 \) if \( x \geq 0 \), -1 otherwise)
   
   (d) \( e^t \), \( \sin t \), \( \cos t \).

3. Let \( x_1 = (a_{11}, \ldots, a_{1n}), \ldots, x_n = (a_{n1}, \ldots, a_{nn}) \) be elements of \( \mathbb{K}^n \) with
   
   \[ |a_{ii}| > \sum_{j=1, j \neq i}^{n} |a_{ji}| \quad \text{for all } i = 1, \ldots, n. \]

   Show that \( x_1, \ldots, x_n \) are linearly independent over \( \mathbb{K} \). (**Hint**: Suppose that \( b_1 x_1 + \cdots + b_n x_n = 0 \). Then we can find \( b'_1, \ldots, b'_n \in \mathbb{K} \) such that \( |b'_i| \leq 1 \) for all \( i \) and \( b'_j = 1 \) for some \( j \). Then what happens to the given condition?)

4. Let \( \lambda_1, \ldots, \lambda_n \) be pairwise distinct elements in a field \( K \). Prove that the vectors \( x_1 := (1, \lambda_1, \lambda_1^2, \ldots, \lambda_1^{n-1}) \), \( \ldots \), \( x_n := (1, \lambda_n, \lambda_n^2, \ldots, \lambda_n^{n-1}) \) in \( K^n \) are linearly independent over \( K \). (**Hint**: Induction on \( n \). Assume the result for \( n - 1 \). Prove the result for \( n \), assume that \( a_1 x_1 + \cdots + a_n x_n = 0 \). Then we have the equations: \( a_1 \lambda_n x'_1 + \cdots + a_n \lambda_n x'_n = 0 \) and \( a_1 \lambda_1 x'_1 + \cdots + a_n \lambda_n x'_n = 0 \), and so \( a_1 (\lambda_n - \lambda_1) x'_1 + \cdots + a_{n-1} (\lambda_n - \lambda_{n-1}) x'_{n-1} = 0 \), where \( x'_i := (1, \lambda_i, \ldots, \lambda_i^{n-2}), i = 1, \ldots, n. \)
5. Let \( \lambda_1, \ldots, \lambda_n \) be pairwise distinct elements in a field \( K \). The vectors \( y_1 := (1,1,\ldots,1), y_2 := (\lambda_1,\lambda_2,\ldots,\lambda_n), \ldots, y_n := (\lambda_1^{n-1},\ldots,\lambda_n^{n-1}) \in K^n \) are linearly independent over \( K \) (and hence is a \( K \)-basis of \( K^n \)).

(Hint: Note that a representation \( b_1y_1 + \cdots + b_ny_n = 0 \) with \( b_1, \ldots, b_n \in K \) is equivalent with the system of equations \( b_1 + b_2\lambda_i + \cdots + b_n\lambda_i^{n-1} = 0 \), \( i = 1, \ldots, n \). Therefore the vectors \( x_i, \ i = 1, \ldots, n \), are solutions of the homogeneous system of linear equations \( b_1z_1 + \cdots + b_nz_n \). Since \( x_1, \ldots, x_n \) is a generating system of \( K^n \), the solution space of this equation is \( K^n \) which is possible only in the case \( b_1 = \cdots = b_n = 0 \).

Another Argument: The equations \( b_1 + b_2\lambda_i + \cdots + b_n\lambda_i^{n-1} = 0 \), \( i = 1, \ldots, n \), mean that the polynomial \( b_1 + b_2X + \cdots + b_nX^{n-1} \in K[X] \) of degree < \( n \) has \( n \) pairwise distinct zeros \( \lambda_1, \ldots, \lambda_n \in K \) and hence \( b_1 = \cdots = b_n = 0 \).)