Linear Algebra: Exercise 2 - to be submitted

Maximum marks: 20

1. Let $V$ be a vector space over a field $K$ where $|K| \geq n$. Further let $V_1, V_2, \ldots, V_n$ be $K$-subspaces of $V$. If $V_i \neq V$ for every $1 \leq i \leq n$ then show that $V_1 \cup V_2 \cdots \cup V_n \neq V$. Show by an example that the condition $|K| \geq n$ is necessary. (Hint: By induction on $n$, assume that $V_1 \cup V_2 \cdots \cup V_{n-1} \neq V$. Choose $x \in V_n$ with $x \notin V_1 \cup \cdots \cup V_{n-1}$ and $y \in V$ with $y \notin V_n$. Now consider the set $\{ax + y \mid a \in K\}$ which has at least $n$ distinct elements.) – (10 marks).

2. Let $K$ be a field of characteristic $\neq 2$, i.e. $1 + 1 \neq 0$ in $K$ and let $a \in K$. Compute the solution set of the following systems of linear equations over $K$:

$$ax_1 + x_2 + x_3 = 1 \quad \quad \quad \quad \quad \quad \quad \quad x_1 + x_2 - x_3 = 1$$
$$x_1 + ax_2 + x_3 = 1 \quad \quad \quad \quad \quad \quad \quad \quad 2x_1 + 3x_2 + ax_3 = 3$$
$$x_1 + x_2 + ax_3 = 1 \quad \quad \quad \quad \quad \quad \quad \quad x_1 + ax_2 + 3x_3 = 2$$

For which $a$ these systems have exactly one solution? – (10 marks)
1. Show that the set of \( m \)-tuples \((b_1, \ldots, b_m) \in K^m\) for which a linear system of equations \( \sum_{j=1}^{n} a_{ij}x_j = b_i, \ i = 1, \ldots, m\), over a field \( K \) has a solution is a \( K \)-subspace of \( K^m \).

2. Let \( K \) be a subfield of the field \( L \) and let \( \sum_{j=1}^{n} a_{ij}x_j = b_i, \ i = 1, \ldots, m \) be a system of linear equations over \( K \). Prove that if this system has a solution \((x_1, \ldots, x_n) \in L^n\), then it also has a solution in \( K^n \).

3. (a) Let \( p_1, p_2, p_3 \) be three distinct prime numbers. Show that the (positive) square-roots \( \sqrt{p_1}, \sqrt{p_2}, \sqrt{p_3} \) are linearly independent over \( \mathbb{Q} \).
   (b) Show that the family \( \{ \ln p \mid p \text{ prime number} \} \) of real numbers is linearly independent over \( \mathbb{Q} \). (Hint: Use the Fundamental Theorem of Arithmetic)

4. Let \( x_1, \ldots, x_n \in V \) be linearly independent (over \( K \)) in a \( K \)-vector space \( V \) and let \( x := \sum_{i=1}^{n} a_i x_i \in V \) with \( a_i \in K \).
   Show that \( x_1 - x, \ldots, x_n - x \) are linearly independent over \( K \) if and only if \( a_1 + \cdots + a_n \neq 1 \).

5. Prove the following:
   (a) The family \( x_1, \ldots, x_n, x_1 + \cdots + x_n \) is linearly dependent over \( K \), but every \( n \) of these vectors are linearly independent over \( K \).
   (b) Show that \( x_1, \ldots, x_n, x \) are linearly independent over \( K \) if and only if \( x_1, \ldots, x_n \) are linearly independent over \( K \) and \( x \notin (x_1 + \cdots + Kx_n) \).
   (c) Show that \( x_1, \ldots, x_n \) is a generating system of \( V \) if and only if \( x_1, \ldots, x_n, x \) is a generating system of \( V \) and \( x \in Kx_1 + \cdots + Kx_n \).