Linear Algebra: Exercise 1 - to be submitted

Maximum marks: 20

In the following questions \( \mathbb{N} \) denotes the set of natural numbers, \( \mathbb{N}^* = \mathbb{N} \setminus \{0\} \), \( \mathbb{R} \) denotes the field of real numbers, \( \mathbb{C} \) denotes the field of complex numbers and \( \mathbb{K} \) is either equal to \( \mathbb{R} \) or \( \mathbb{C} \).

1. Let \( \mathcal{N} \subseteq \mathbb{N} \) be a submonoid \( \neq 0 \) of the additive monoid \( \mathbb{N} \) with the following property: If \( a, b \in \mathcal{N} \) and \( a \leq b \) then \( b - a \in \mathcal{N} \). Show that \( \mathcal{N} = \mathcal{N}_n = \{an \mid a \in \mathbb{N}\} \) for some uniquely determined \( n \in \mathbb{N}^* := \mathbb{N} \cap \mathbb{N}^* \). – (10 marks)

2. A function \( f : \mathbb{R} \rightarrow \mathbb{K} \) is called even (respectively, odd) if \( f(-x) = f(x) \) (respectively, \( f(-x) = -f(x) \)) for all \( x \in \mathbb{R} \). For example, the sine \( \sin : \mathbb{R} \rightarrow \mathbb{R} \) (respectively, cosine \( \cos : \mathbb{R} \rightarrow \mathbb{R} \)) function is an odd (respectively, even) function.

(a) Show that \( \mathbb{K}^\mathbb{R} \) is a vector space. – (4 marks)

(b) Show that the set \( W_{\text{even}} \) (resp. \( W_{\text{odd}} \)) of all even (resp. odd) functions \( \mathbb{R} \rightarrow \mathbb{K} \) is a \( \mathbb{K} \)-subspace of \( \mathbb{K}^\mathbb{R} \). – (6 marks)
Practice Problems 1 - not to be submitted

In the following questions \( \mathbb{R} \) denotes the field of real numbers and \( \mathbb{C} \) denotes the field of complex numbers.

1. For \( a, b \in \mathbb{R} \), let \( f_{a,b} : \mathbb{R} \to \mathbb{R} \) be the function defined by \( f_{a,b}(x) := ax + b \), \( x \in \mathbb{R} \). Show that \( A := \{ f_{a,b} \mid a, b \in \mathbb{R}, a \neq 0 \} \) with the composition of functions as a binary operation is a non-commutative group.

2. Let \( n \) be a positive integer greater than 2. Show that \((\mathbb{Z}_n, +, n, \cdot n)\) is not a field when \( n \) is not a prime number.

3. Let \( V \) be a vector space over a field \( K \). For arbitrary elements \( p, q \in K \) and arbitrary vectors \( x, y \in V \), prove that
   
   (a) \( 0 \cdot x = p \cdot 0 = 0 \).
   (b) \( p(-x) = (-p)x = -(px) \).
   (c) \( (-p)(-x) = px \).
   (d) \( p(x - y) = px - py \) and \( (p - q)x = px - qx \).

4. For subspaces \( U, U', W, W' \) of a vector space \( V \) over a field \( K \), show that:

   (a) \( U + (U' \cap W) \subseteq (U + U') \cap (U + W) \).
   (b) \( U \cap (U' + W) \supseteq (U \cap U') + (U \cap W) \).
   (c) If \( U \subseteq U' \), then \( U + (U' \cap W) = U' \cap (U + W) \).
   (d) If \( U \cap W = U' \cap W' \), then \( U = (U + (W \cap U')) \cap (U + (W \cap W')) \).

5. Let \( V \) be a vector space over a field \( K \) and let \( X \) be any set with a bijection \( f : X \to V \). Then \( X \) has a \( K \)-vector space structure with \( f^{-1}(0) \) as the zero element and for \( a \in K, x, y \in X \), \( x + y := f^{-1}(f(x) + f(y)) \) and \( a \cdot x := f^{-1}(af(x)) \).

6. Let \( X \) be a non-empty set and \( D \subset X \). Consider the set of functions \( S := \{ f : X \to \mathbb{C} \mid f(x) = 0, \forall x \in D \} \). Show that \( S \) is a \( \mathbb{C} \)-subspace of the vector space \( \mathbb{C}^X \) of all \( \mathbb{C} \)-valued functions on \( X \).