Assignment 7


1. Let \( f, g : X \rightarrow \mathbb{R} \) be two continuous functions on a compact set \( X \). Prove that \( f + g \) is a continuous function on \( X \).

2. Let \( A \) be an open set in \( \mathbb{R} \). Prove that \( A \cap \mathbb{Q} \) is a dense subset of \( A \).

3. For any \( x \in \mathbb{R} \), let \( f_n(x) = \frac{1}{n} \) for all \( n \geq 1 \). Prove that \( f_n \) converges uniformly to the function \( f(x) = 0 \) on \( \mathbb{R} \).

4. Let \( A \) be a bounded set in \( \mathbb{R} \). Prove that \( A \) has a maximum and a minimum.

5. Find the volume of the solid obtained by revolving the region bounded by the curves \( y = x^2 \) and \( y = 4 - x^2 \) about the \( y \)-axis.

6. Evaluate the definite integral \( \int_0^\infty e^{-x^2} \, dx \).

7. Let \( f : [0, 1] \rightarrow \mathbb{R} \) be a continuous function. Prove that there exists a \( c \in [0, 1] \) such that\( f(c) = \frac{1}{2} \int_0^1 f(x) \, dx \).

8. Let \( f : [a, b] \rightarrow \mathbb{R} \) be a continuous function. Prove that there exists a \( c \in (a, b) \) such that \( f(c) = \frac{1}{b-a} \int_a^b f(x) \, dx \).

9. Let \( A \) be a nonempty, compact subset of \( \mathbb{R} \). Prove that \( A \) has a maximum and a minimum.

10. Let \( f : [a, b] \rightarrow \mathbb{R} \) be a continuous function. Prove that there exists a \( c \in (a, b) \) such that \( f(c) = \frac{1}{b-a} \int_a^b f(x) \, dx \).

11. Let \( X \) be a compact metric space. Prove that every continuous function \( f : X \rightarrow \mathbb{R} \) is uniformly continuous.

12. Let \( f : [a, b] \rightarrow \mathbb{R} \) be a continuous function. Prove that there exists a \( c \in (a, b) \) such that \( f(c) = \frac{1}{b-a} \int_a^b f(x) \, dx \).