Assignment 5

Due on 2020-03-04, 23:59 IST.

Part A: Linear Algebra

Unit 6 - Week 5

Course outline

Here are notes on the topics covered in this week:

1. Inner product spaces
2. Orthogonality
3. Change of basis
4. Matrix representations
5. Linear transformations

Assignment

Problem 1: Consider the linear transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $T(x, y) = (2x - y, x + 2y)$.

(a) Find the matrix representation of $T$ with respect to the standard basis.

(b) Find the kernel and the image of $T$.

Problem 2: Let $V$ be a vector space over $\mathbb{F}$ and let $f: V \rightarrow V$ be a linear transformation. Suppose $f^2 = f$. Prove that $f$ is diagonalizable.

Problem 3: Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear transformation defined by $T(x, y, z) = (x - 2y + 3z, 2x + y - z, 3x - y + 2z)$. Find a basis for the kernel of $T$.

Problem 4: Let $W$ be a subspace of $\mathbb{R}^4$ and let $f: W \rightarrow W$ be a linear transformation. Suppose $f^3 = 0$ and $f^2 \neq 0$. Prove that $W$ is not a direct sum of two proper subspaces.

Problem 5: Let $V$ be a finite-dimensional vector space and let $f: V \rightarrow V$ be a linear transformation. Suppose $f^n = 0$ for some positive integer $n$. Prove that $V$ is a direct sum of the kernels of $f, f^2, \ldots, f^{n-1}$.

Part B: Numerical Analysis

Problem 6: Consider the system of linear equations $Ax = b$ where $A$ is a $3 \times 3$ matrix and $b$ is a $3 \times 1$ column vector. Use Gaussian elimination to find the solution of the system.

Problem 7: Let $f(x) = x^3 - 2x^2 + 3x - 4$. Use the Newton-Raphson method to find a root of $f(x)$ accurate to within 0.01.

Part C: Probability Theory

Problem 8: Consider a random variable $X$ with probability mass function $P(X = k) = \frac{1}{2^k}$ for $k = 0, 1, 2, \ldots$. Find the expected value of $X$.

Problem 9: Let $A$ be a $2 \times 2$ matrix with eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 3$. Find the determinant of $A$.