Unit 3 - Week 2

Assignment 2

Due on 2020-02-12, 23:59 IST.

Note: All rings considered in this course are commutative with unity.

1. Let $\mathbb{Z}$ denote the ring of integers. Let $R = \mathbb{Z}[\sqrt{2}]$ be the subring of the field of real numbers defined as $R = \{a + b\sqrt{2} | a, b \in \mathbb{Z}\}$ with its usual addition and multiplication of real numbers. Determine which of the following are ideals in the ring $R$.

- $\{0\}$ is an ideal.
- $\{(a + b\sqrt{2}) | a, b \in \mathbb{Z}, a + b = 0\}$ is an ideal.
- $\{(a + b\sqrt{2}) | a, b \in \mathbb{Z}, 4\text{ divides } a, 4\text{ divides } b\}$ is an ideal.

2. Let $S$ denote the field of real numbers. Determine which of the following are ideals in the polynomial ring $R = \mathbb{R}[x]$.

- $\{0\}$ is an ideal.
- The set of all polynomials of all degrees is not an ideal.
- The set of all polynomials of degree at least 10, along with the zero polynomial, is an ideal.
- The set of all polynomials which have $2$ as a root is not an ideal.

3. Select ALL true statements. (Note that, by definition, any ring homomorphism $R \rightarrow R'$ sends $1 \to 1$. $Z$ denotes the ring of integers and $Q$ denote the field of rational numbers.

- There are at least two ring homomorphisms $Z \rightarrow Q$.
- There exists exactly one ring homomorphism $Z \rightarrow Q$.
- If $f$ is any nonzero ring homomorphism, then $f(1)$ is the identity.
- If $f$ is any nonzero ring homomorphism, then $f(1)$ is injective.

4. Let $R$ be a ring. An element $x \in R$ is called a nilpotent if $x^n = 0$ for some positive integer $n$. Let $f : R \rightarrow R'$ be a ring homomorphism. Let $f(x) = 0$. Choose all the correct statements.

- $x$ is a unit, then $f(x)$ is a unit.
- $f(x)$ is a unit, then $x$ is a unit.
- $f(x)$ is nilpotent, then $x$ is nilpotent.
- $f(x)$ is nilpotent, then $x$ is nilpotent.

5. Let $R$ be an arbitrary ring. Choose which of the following statements are true always.

- The set of units in $R$ forms an ideal.
- The set of zero divisors in $R$ along with the zero element forms an ideal of $R$.
- The set of nilpotent elements in $X$ forms an ideal of $R$.

6. Let $I, J$ be two ideals in a ring $R$. Define the join of ideals as follows:

$$ I \lor J = \left\{ a + b | a \in I, b \in J \right\} $$

Determine which of the following statements are always true.

- If $I, J$ are ideals in $R$, then $I \lor J$ is an ideal.
- If $I, J$ are ideals in $R$, then $I \lor J$ is a subring.
- The union $I \cup J$ is a subset of $I \lor J$.

7. Let $I, J$ be two ideals in a ring $R$. Define the meet of ideals as follows:

$$ I \land J = \left\{ a \in R | a \in I, a \in J \right\} $$

Determine which of the following statements are always true.

- If $I, J$ are ideals in $R$, then $I \land J$ is an ideal.
- If $I, J$ are ideals in $R$, then $I \land J$ is a subring.
- The union $I \lor J$ is a subset of $I \land J$.