Assignment - 9

(1) Find the Solution of the initial Value problem

$$u_{tt} - 3u_{xx} = 0, \quad x \in \mathbb{R}, t > 0$$
$$u(x,0) = x\sin x, \quad x \in \mathbb{R}$$
$$u_t(x,0) = \cos 2x, \quad x \in \mathbb{R}.$$

Solution:

By D' Alembert's formula,

$$u(x,t) = \frac{1}{2}\left[(x-\sqrt{3}t)\sin(x-\sqrt{3}t) + (x+\sqrt{3}t)\sin(x+\sqrt{3}t)\right] + \frac{1}{2\sqrt{3}}\int_{x-\sqrt{3}t}^{x+\sqrt{3}t} \cos 2u \, du$$

$$= \frac{1}{2}\left[(x-\sqrt{3}t)\sin(x-\sqrt{3}t) + (x+\sqrt{3}t)\sin(x+\sqrt{3}t)\right] + \frac{1}{4\sqrt{3}}\left[\sin(x+\sqrt{3}t) - \sin(x+\sqrt{3}t)\right].$$

(2) Find the Solution of the initial boundary value problem

$$u_{tt} - 4u_{xx} = 0, \quad x > 0, t > 0$$

Initial
$$\begin{cases} u(x,0) = \sin^2 x, & x \geq 0 \\ u_t(x,0) = 0, & x \geq 0. \end{cases}$$

BC $\{ u(0,t) = 0, \; t \geq 0.$

Solution: By using D' Alembert's formula, we obtain,

$$u(x,t) = \begin{cases} \frac{1}{2}\left[\sin^2(x+2t) - \sin^2(x-2t)\right], & 0 \leq x < 2t \\ \frac{1}{2}\left[\sin^2(x+2t) + \sin^2(x-2t)\right], & x > 2t \end{cases}$$

③ Find the solution of the initial boundary value problem,

$$u_{tt} - 4u_{xx} = 0 \quad x>0,\, t>0$$

$$u(x,0) = \sin x \;,\; x \geq 0$$

$$u_t(x,0) = x^2 \;,\; x \geq 0.$$

$$u(0,t) = 0 \;,\; t \geq 0.$$

Solution:

Again by the use of D' Alembert's formulae,

$$u(x,t) = \begin{cases} \frac{1}{2}\left[-\sin(2t-x) + \sin(2t+x)\right] + \frac{1}{4}\left(\frac{(x+2t)^3}{3} - \frac{(2t-x)^3}{3}\right), & 0<x<2t \\ \frac{1}{2}\left[\sin(x+2t) + \sin(x-2t)\right] + \frac{1}{4}\left(\frac{(x+2t)^3}{3} - \frac{(x-2t)^3}{3}\right), & x>2t. \end{cases}$$

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④ Find the solution of the initial boundary value problem.

$$u_{tt} - u_{xx} = 0 \;,\; x>0,\, t>0$$

$$u(x,0) = \cos\left(\frac{\pi x}{2}\right),\; x \geq 0$$

$$u_t(x,0) = 0 \;,\; x \geq 0$$

$$u_x(0,t) = 0 \;,\; t \geq 0.$$

Solution:

(using the formula obtained in the lecture).

the solution is

$$u(x,t) = \begin{cases} \frac{1}{2}\left[\cos\left(\frac{\pi}{2}(t-x)\right) + \cos\left(\frac{\pi}{2}(t+x)\right)\right], & 0<x\leq t \\ \frac{1}{2}\left[\cos\frac{\pi}{2}(x-t) + \cos\frac{\pi}{2}(x+t)\right], & x>t \end{cases}$$

$$u(x,t) = \cos\left(\frac{\pi x}{2}\right)\cos\left(\frac{\pi}{2}t\right), \quad t>0,\, x>0.$$

⑤ Consider the pde $u_{tt}+\alpha u_{tx}+\beta u_{xx}=0$, $x\in\mathbb{R}, t>0$, where $\alpha,\beta\in\mathbb{R}$ such that $\lambda^2+\alpha\lambda+\beta=0$, has two distinct roots $c_1, c_2\in\mathbb{R}$. Then, use the transformation variables $\xi = x+c_1 t$ and $\eta = x+c_2 t$ to reduce the above pde to $u_{\xi\eta}=0$.

<u>Solution:</u> first note that $\alpha=-(c_1+c_2)$; $\beta = c_1 . c_2$.

$$u_t = u_\xi \xi_t + u_\eta \eta_t$$

$$= c_1 u_\xi + c_2 u_\eta$$

$$u_{tt} = c_1^2 u_{\xi\xi} + 2c_1c_2 u_{\xi\eta} + c_2^2 u_{\eta\eta}$$

$$u_x = u_\xi + u_\eta$$

$$u_{xt} = c_1(u_{\xi\xi}+u_{\xi\eta}) + c_2(u_{\xi\eta}+u_{\eta\eta})$$

$$= c_1 u_{\xi\xi} + (c_1+c_2)u_{\xi\eta} + c_2 u_{\eta\eta}.$$

$$u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}.$$

now,

$$u_{tt}+\alpha u_{tx}+\beta u_{xx} = c_1^2 u_{\xi\xi} + 2c_1c_2 u_{\xi\eta} + c_2^2 u_{\eta\eta} - (c_1+c_2)\begin{bmatrix} c_1 u_{\xi\xi} + c_2 u_{\eta\eta} \\ + (c_1+c_2) u_{\xi\eta}\end{bmatrix}$$

$$+ c_1c_2(u_{\xi\xi}+2u_{\xi\eta}+u_{\eta\eta})$$

$$= 2c_1c_2 u_{\xi\eta} - \cancel{c_1c_2 u_{\xi\xi}} - \cancel{c_1c_2 u_{\eta\eta}} - (c_1^2+c_2^2+2c_1c_2)u_{\xi\eta}$$

$$+ \cancel{c_1c_2 u_{\xi\xi}} + \cancel{c_1c_2 u_{\eta\eta}} + 2c_1c_2 u_{\xi\eta}$$

$$= -(c_1-c_2)^2 u_{\xi\eta}$$

Therefore $(c_1-c_2)^2 u_{\xi\eta} = 0$, Since $c_1 \neq c_2$ (distinct)

implies $u_{\xi\eta}=0$