1. Classify the partial differential equations:

(i) \( \frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial y} - 3\frac{\partial^2 u}{\partial y^2} = 0 \)

Since \( 4 + 4(8)(3) > 0 \),

(i) is of hyperbolic type.

(ii) \( \frac{\partial^2 u}{\partial x^2} + 8\frac{\partial^2 u}{\partial x \partial y} + 5\frac{\partial^2 u}{\partial y^2} = 0 \)

Since, \( 4 - 4(5)(0) < 0 \), (ii) is elliptic.

(iii) \( 4\frac{\partial^2 u}{\partial x^2} - 12\frac{\partial^2 u}{\partial x \partial y} + 9\frac{\partial^2 u}{\partial y^2} = 0 \)

Since, \( (-12)^2 - 4(4)(9) = 0 \), (ii) is parabolic.

2. Classify the partial differential equations:

(i) \( \frac{\partial^2 u}{\partial x^2} + (x-1)\frac{\partial^2 u}{\partial x \partial y} - 2x\frac{\partial u}{\partial x} + 3xy\frac{\partial u}{\partial y} + 2u = \sin x \)

Consider \( \Delta = (x-1)^2 - 4(4) = (x-1)^2 - 4 \)

\( \bullet \) (i) is parabolic if \( \Delta > 0 \Leftrightarrow (x-1)^2 > 4 \Leftrightarrow x > 3 \) or \( x < -1 \)

\( \bullet \) (i) is elliptic if \( \Delta < 0 \Leftrightarrow (x-1)^2 < 4 \Leftrightarrow -1 < x < 3 \)

\( \bullet \) (i) is hyperbolic if \( \Delta < 0 \Leftrightarrow (x-1)^2 > 4 \Leftrightarrow x > 3 \) or \( x < -1 \)

\( \bullet \) (i) is parabolic if \( \Delta = 0 \Leftrightarrow (x-1)^2 = 4 \Leftrightarrow x = 3 \) and \( x = -1 \)

HYPERBOLIC $x > 3$

ELLITPIC $-1 < x < 3$

PARABOLIC $x = 3$

HYPERBOLIC $x < -1$
2(iii) \( u_{xx} + 2x u_{xy} + (1-y^2) u_{yy} = 0 \)

Let \( \Delta = (2x)^2 - 4(1-y^2) \).

- (ii) is \underline{parabolic} if \( \Delta = 0 \) \( \implies \ (2x)^2 + 4y^2 = 4 \\ \\ \\ \\ \ \ \ \ \ \implies x^2 + y^2 = 1 \)
  i.e., the pde is parabolic if \((x, y)\) satisfies \(x^2 + y^2 = 1\).

- (ii) is \underline{elliptic} if \( \Delta < 0 \) \( \implies \ (2x)^2 - 4y^2 + 4 \\ \\ \implies x^2 + y^2 < 1 \).

- (iii) is \underline{hyperbolic} if \( \Delta > 0 \) \( \implies x^2 + y^2 > 1 \).

2(iii') \( (1+y^2) u_{xx} + (1+y^2) u_{yy} = 0 \)

Let \( \Delta = -4(1+x^2) (1+y^2) \).

(iii) is \underline{parabolic} if \( (x^2 + 1)(y^2 + 1) = 0 \) \( \implies x \in \mathbb{R}, \ y \in \mathbb{R} \)

i.e., (iii) is nowhere parabolic.

- Observe that \( \Delta \leq 0 \) for all \((x, y) \in \mathbb{R}^2\).

Therefore, 2(iii) is elliptic pde.
3) Reduce the PDE \( 2u_{xx} - 2u_{xy} + 5u_{yy} = 0 \) into a normal form.

Solution: First, observe that given PDE is elliptic type.

The characteristic equations are

\[
\frac{dy}{dx} = \frac{-2 \pm \sqrt{4 - 4.25}}{2} \quad \text{and} \quad \frac{dy}{dx} = \frac{-2 \sqrt{4 - 4.25}}{2}
\]

\[\Rightarrow \frac{dy}{dx} = \int \left( \frac{-1}{2} + \frac{1}{2}i \sqrt{2} \right) \, dx + c_1 \quad \text{and} \quad \frac{dy}{dx} = \int \left( \frac{-1}{2} - \frac{1}{2}i \sqrt{2} \right) \, dx + c_2.\]

\[\Rightarrow 2y + x - \frac{i}{\sqrt{2}}x = c_1 \quad \text{and} \quad 2y + x + \frac{i}{\sqrt{2}}x = c_2.
\]

Now, choose the transformations. \( \xi = 2y + x \quad \text{and} \quad \eta = \frac{3}{2}x \).

So that,

\[u_x = u_{\xi} + 3u_{\eta},\]

\[u_{xx} = u_{\xi\xi} + 3u_{\eta\xi} + 3(u_{\xi\eta} + 3u_{\eta\eta}),\]

\[u_y = 2u_{\xi}, \quad u_{yy} = 4u_{\xi\xi},\]

\[u_{xy} = 2(u_{\xi\eta} + 3u_{\eta\xi}).\]

Therefore, the given PDE becomes,

\[2u_{\xi\xi} + 12u_{\xi\eta} + 18u_{\eta\eta} - 4u_{\xi} - 12u_{\eta} + 20u_{\xi\eta} = 0\]

\[\Rightarrow u_{\eta\eta} + u_{\xi} = 0,\]

which is a normal form.

4) Reduce the PDE \( u_{xx} - 2u_{xy} = 0 \) into a normal form.

Solution: Observe that the given PDE is hyperbolic type.

The characteristic equations are

\[\frac{dy}{dx} = \frac{-2 \pm 2}{2}, \quad \frac{dy}{dx} = \frac{-2 - 2}{2}.
\]

\[\Rightarrow y = c_1 \quad \text{and} \quad y + 2x = c_2.\]
Now, choose the transformation \( \xi = y \); \( \eta = y + 2x \).

So that, \( U_\eta = 2U_\xi \) & \( U_{\xi \eta} = 4U_\eta \).

Therefore, the given pde becomes, \( 4U_{\eta \eta} - 2(U_{\eta \xi} + 2U_{\xi \xi}) = 0 \).

ie., \( U_{\xi \eta} = 0 \) is the required normal form.

5. Reduce the pde \( U_{\xi \xi} + (2x+3)U_{\xi \eta} + 6U_{\eta \eta} = 0 \) into a normal form.

**Solution:** First check that \( (2x+3)^2 - 4(6)x = (2x-3)^2 > 0 \) if \( x > \frac{3}{2} \).

That is, the given pde is hyperbolic if \( x > \frac{3}{2} \), and it is parabolic at \( x = \frac{3}{2} \).

Let \( x > \frac{3}{2} \): the corresponding characteristic equations are

\[
\frac{dy}{dx} = 3 \quad \text{and} \quad \frac{dy}{dx} = -2x = 0.
\]

\( \Rightarrow y = 3x \) & \( y - x^2 = C_2 \) are characteristic curves.

Now, choose the transformation \( \xi = y - 3x \) & \( \eta = y - x^2 \).

So that \( U_\eta = -2U_\xi - 2xU_\eta \)

\[ U_{\xi \eta} = \left( \frac{-2 \xi^2 - 2x^2}{2 \xi} \right)(-2U_\xi - 2xU_\eta) \]

We know that,

\[
\begin{align*}
U_{\xi \xi} &= U_{\xi \xi} \xi^2 + 2U_{\xi \eta} \xi \eta + U_{\eta \eta} \eta^2 + U_\eta \xi \eta + U_\eta \eta \xi + U_\xi \eta \xi \\
U_{\xi \eta} &= U_{\xi \eta} \xi^2 + 2U_{\xi \xi} \xi \eta + U_{\eta \eta} \eta^2 + U_\eta \xi \eta + U_\eta \eta \xi + U_\xi \eta \xi \\
\end{align*}
\]

and \( U_{\eta \eta} = U_{\eta \eta} \xi^2 + 2U_{\eta \xi} \xi \eta + U_{\xi \xi} \eta^2 + U_\xi \eta \xi + U_\xi \eta \eta + U_\eta \xi \eta + U_\eta \eta \xi + U_\xi \eta \xi \)
Reduce the PDE \((\sin^2 x) \frac{\partial^2 u}{\partial x^2} + (\sin^2 x) \frac{\partial u}{\partial x} + (\cos^2 x) \frac{\partial u}{\partial y} = 0\) into a normal form.

**Solution:** Note that \((\sin^2 x)^2 - 4 \sin^2 x \cos^2 x = 0\), that is, the given PDE is parabolic type.

To find the characteristic curves, set \(\frac{dy}{dx} = \frac{\sin^2 x}{2 \sin^2 x}\).

This implies that \(y = \ln |\sin x| + C\).

Now, choose the transformation \(\xi = y - \ln |\sin x|\) and \(\eta = x\).

Clearly, the Jacobian \(\frac{\partial (\xi, \eta)}{\partial (x, y)} = 0\).

Now compute: \(U_x = U_\xi (\cot x) + U_\eta\).

As \(U_{xx} = U_{\xi \xi} (\cot x)^2 + 2 U_{\xi \eta} (\cot x) + U_{\eta \eta} + U_{\eta \xi} (\cos^2 x) + 0\).

\(U_{xy} = U_{\xi \eta} (0) + 0 + 0 + 0 = 0\).

\(U_{yx} = U_{\eta \xi} (\cot x) + U_{\eta \eta} + 0 + 0 + 0\).

Therefore, the PDE becomes:

\[ U_{\xi \xi} \left( \cot^2 x \cdot \sin^2 x + \cos^2 x - \sin^2 x \cdot \cot x \right) + U_{\xi \eta} \left( \sin^2 x (\cot x) + \sin^2 x (1) \right) + \sin^2 \theta \cdot U_{\eta \eta} + U_{\xi \xi} = 0 \]

\[ \Rightarrow \sin^2 \theta \cdot u_{\xi \xi} + u_{\xi \xi} = 0 \]

is a normal form of the given PDE.
Reduce the pde \( U_{xx} - 2\sin x U_{xy} - \cos^2 x U_{yy} - \cos x U_y = 0 \).

**Solution:**

Since \((-2\sin x)^2 - 4(1)(-\cos^2 x) = 4(\cos^2 x + \sin^2 x) = 4 > 0\),

so, the given pde is hyperbolic type.

The characteristic equations are

\[
\frac{dx}{\sin^2 x + 1} = \frac{dy}{\sin x - 1} = 0
\]

Now, set the transformations

\[
\xi = y - \cos x + x \quad \eta = y - \cos x - x
\]

then, \( U_x = (1 + \sin x) U_\xi + (\sin x - 1) U_\eta \),

\( U_y = U_\xi + U_\eta \).

\( U_{xx} = U_{\xi\xi}(1 + \sin x)^2 + 2 U_{\xi\eta}(\sin x - 1) + U_{\eta\eta}(\sin x - 1) + u_\xi^2(\cos x) + u_\eta(\cos x) \)

\( U_{xy} = U_{\xi\xi} + 2 U_{\xi\eta} + U_{\eta\eta} \to 0 \).

\( U_{yy} = U_{\xi\xi}(1 + \sin x) + U_{\xi\eta}(1 + \sin x + \sin x - 1) + U_{\eta\eta} + o(0) \).

Therefore, the differential equation becomes,

\[
U_{\xi\xi}((1 + \sin x)^2 - 2\sin x (1 + \sin x) - \cos^2 x) + U_{\xi\eta}(2(\sin^2 x - 1) - 2\sin x (2\sin x) - 2\cos^2 x) + U_{\eta\eta}((\sin x - 1)^2 - 2\sin x (\cos^2 x) + \cos x (U_\xi + U_\eta)) = 0
\]

\( \Rightarrow \) please complete the problem by taking care of mistakes.