1. Find the singular points and classify them for the equation \( x^3 (x-2) y'' + x^3 y' + 6y = 0 \).

**Solution:** Since \( x^3 (x-2) = 0 \) \( \Rightarrow \) \( x = 0 \) or \( x = 2 \). That is \( x = 0 \) and \( x = 2 \) are singular points for given differential equation.

If we rewrite the DE in standard form, then

\[
\frac{1}{p(x)} y'' + \frac{1}{q(x)} y' + \frac{6}{r(x)} y = 0, \quad x \neq 0, x \neq 2.
\]

Now, the limits,

\[
\lim_{x \to 0} x p(x) = \lim_{x \to 0} \frac{x}{x-2} = 0 < \infty, \text{ (finite)}
\]

\[
\lim_{x \to 0} x^2 q(x) = \lim_{x \to 0} \frac{1}{x(x-2)} = \infty, \text{ (not finite)}
\]

Therefore, \( x = 0 \) is irregular singular point.

And

\[
\lim_{x \to 2} (x-2) p(x) = \lim_{x \to 2} 1 = 1 < \infty, \text{ finite}
\]

\[
\lim_{x \to 2} (x-2)^2 q(x) = \lim_{x \to 2} \frac{(x-2) 6}{x^3} = 0 < \infty, \text{ finite}
\]

Therefore, \( x = 2 \) is a regular singular point.
Find the power series solution for the differential equation \( y'' + xy' + (1+x^2)y = 0 \rightarrow (I) \).

Solution: clearly \( x=0 \), is a regular point for (I).

We look for a solution of the form \( y(x) = \sum_{n=0}^{\infty} C_n x^n \).

Suppose such \( y(x) \) solves (I), then,

\[
\sum_{n=0}^{\infty} n(n-1) C_n x^{n-2} + \sum_{n=0}^{\infty} n C_n x^{n-1} + \sum_{n=0}^{\infty} C_n x^n + \sum_{n=0}^{\infty} x^{n+2} C_n x^n = 0.
\]

\[
\Rightarrow \sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{n+3} C_{n+2} x^n + \sum_{n=0}^{\infty} (n+1) C_n x^n + \sum_{n=2}^{\infty} C_{n-2} x^n = 0.
\]

\[
\Rightarrow \left( 2C_2 + 3C_3 x \right) + \left( C_0 + 2C_1 x \right) + \sum_{n=2}^{\infty} \left[ \frac{(n+2)(n+1)}{n+3} C_{n+2} + (n+1) C_n + C_{n-2} \right] x^n = 0.
\]

\[
\Rightarrow C_2 = -\frac{1}{2} C_0, \quad C_3 = -\frac{2}{6} C_1, \quad \text{etc.}
\]

\[
C_{n+2} = -\frac{(n+1) C_n - C_{n-2}}{(n+2)(n+1)}, \quad n = 2, 3, \ldots
\]

When \( n=2 \):

\[
C_4 = -\frac{3C_2 - C_0}{4 \cdot 3} = \frac{1}{4!} C_0.
\]

When \( n=3 \):

\[
C_5 = -\frac{4C_3 - C_1}{5 \cdot 4} = \frac{1}{5!} \left[ 4 \cdot \frac{2}{6} - \frac{2}{3!} \right] C_0 = \frac{2}{5!} C_0.
\]

Therefore, the solution can be written as

\[
y(x) = C_0 \left[ 1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \ldots \right] + C \left[ x - \frac{2}{3!} x^3 + \frac{2}{5!} x^5 - \ldots \right].
\]
3. Solve the initial value problem \[ (x^2 - 1)y'' + 3xy' + 2y = 0, \]
with \( y(0) = 4 \) and \( y'(0) = 6 \).

**Solution:**

1. \( x = 0 \) is a regular point for the given DE (IVP).
2. We look for \( y(x) = \sum_{n=0}^\infty C_n x^n \). Then,
   \[
   \sum_{n=0}^\infty C_n (n+2)(n+1) x^n = \sum_{n=0}^\infty C_{n+2} x^{n+2} + 3 \sum_{n=0}^\infty n C_n x^n + \sum_{n=0}^\infty C_n x^{n+1} = 0
   \]
3. \( \Rightarrow -2C_2 - 6C_3 x + 3C_1 x + C_0 x + \sum_{n=2}^{\infty} \left[ \frac{-(n+2)(n+1)C_{n+2}}{(n+1)(n+2)} + \frac{(n(n-1)+3n)C_n}{C_n-1} \right] x^n = 0 \)
4. \( \Rightarrow C_2 = -6; C_3 = \frac{1}{2} C_1 + \frac{C_0}{3} \); \( C_{n+2} = \frac{(n^2 + 2n)C_n + C_{n-1}}{(n+1)(n+2)} \), \( n = 2, 3, \ldots \)
5. Since \( y(0) = 4 \) \( \Rightarrow C_0 = 4 \).
6. \( y'(0) = 6 \) \( \Rightarrow C_1 = 6. \)
7. Therefore, \( C_3 = \frac{6}{2} + \frac{4}{6} = \frac{11}{3} \),
   \[ C_4 = \frac{(4+4)C_2 + C_1}{3 \cdot 4} = \frac{1}{2}, \]
   \[ C_5 = \frac{(9+6)C_3 + C_2}{4 \cdot 5} = \frac{11}{4}, \ldots \]
8. Hence, the solution is:
   \[ y(x) = 4 + 6x + \frac{11}{3} x^2 + \frac{1}{2} x^4 + \frac{11}{4} x^5 + \ldots \]
4. Find the power series solution around the point \( x = 1 \) for the differential equation \( x^2 y'' + xy' + y = 0 \).

**Solution:** Since \( x = 1 \) is a regular point for the DE, we look for a solution of the form \( y(x) = \sum_{n=0}^{\infty} C_n (x-1)^n \).

Set \( t = x - 1 \) and rewrite the differential equation equivalently as,

\[
(t+1)^2 \frac{d^2 y}{dt^2} + (t+1) \frac{dy}{dt} + y = 0.
\]

Therefore,

\[
(t^2 + t + 1) \sum_{n=0}^{\infty} \frac{n(n-1)C_n}{t^{n-2}} + (t+1) \sum_{n=0}^{\infty} nC_n t^{n-1} + \sum_{n=0}^{\infty} C_n t^n = 0.
\]

(Same as)

\[
\sum_{n=0}^{\infty} \frac{(n+2)(n+1)C_{n+2}}{n(n-1)} t^n + \sum_{n=0}^{\infty} \frac{n(n+1)(n+2)C_{n+1}}{n} t^n + \sum_{n=0}^{\infty} C_n t^n = 0.
\]

Now, set the coefficients of \( t^n \) to zero.

For \( n = 2 \):

\[
C_2 = -\frac{9}{2} - \frac{C_0}{2}.
\]

For \( n = 1 \):

\[
C_3 = -\frac{1}{6} \left[ 6C_2 + 2C_1 \right] = -\frac{C_0}{2} + \frac{C_1}{6}.
\]

For \( n = 2 \):

\[
C_4 = -\frac{1}{4} \left[ 15C_3 + 5C_2 \right] = \frac{1}{12} \left[ 15 \left( -\frac{C_0}{2} + \frac{C_1}{6} \right) + \frac{9}{2} (x-1)^4 \right] = -\frac{9}{12} C_0.
\]

Therefore the general solution is,

\[
y(x) = C_0 \left[ 1 - \frac{(x-1)^2}{2} + \frac{1}{12} (x-1)^3 - \frac{9}{12} (x-1)^4 + \ldots \right]
\]

\[
+ C_1 \left[ x + \frac{1}{6} (x-1)^3 + \ldots \right]
\]
5. Find a polynomial approximation of fourth degree to the solution of the equation \((1+2x)y'' + y' + y = 0\), \(y(0) = 0\), \(y'(0) = 1\).

**Solution:** Since \(x = 0\) is a regular point for the given DE, a power-series solution exists in the neighborhood of \(x = 0\). Suppose \(y(x) = \sum_{n=0}^{\infty} c_n x^n\). Then, by initial conditions, we get:

\[c_0 = 0, \quad c_1 = 1\]

and by substituting \(y(x)\) into the differential equation, we get:

\[(1+2x) \left[ 2c_2 + 32c_3 x + 432c_4 x^2 \right] - \left[ c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 \right] + \left[ c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 \right] = 0.

By equating the coefficients of \(x^n\):

\(\Rightarrow\) (i) \(2c_2 - c_1 = 0 \Rightarrow c_2 = \frac{1}{2}\)

(ii) \(4c_2 + 6c_3 + 2c_2 + c_1 = 0 \Rightarrow c_3 = \frac{1}{3}\)

(iii) \(12c_4 + 12c_3 - 2c_2 + c_2 = 0 \Rightarrow c_4 = \frac{5}{84}\)

Therefore, the fourth degree polynomial approximation is:

\[y(x) \approx x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{5}{84} x^4\]
1. Find the polynomial approximation of fourth degree to the solution of the equation \( y'' + xy' + (1+x)y = 0 \), \( y(0) = 1 \), \( y(0) = 0 \).

**Solution:** Since \( x = 0 \) is a regular point for the given DE, so a power-series solution exists in a neighborhood of \( x = 0 \).

Suppose \( y(x) = \sum_{n=0}^{4} c_n x^n \). Then, by initial conditions \( c_0 = 1 \) and \( c_1 = 0 \), and by substituting \( y(x) \) into its differential equation, we get

\[
\left[2c_2 + 6c_3 x + 12c_4 x^2\right] + \alpha \left[c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3\right] + (1+x) \left[c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4\right] = 0.
\]

By equating the coefficients of \( x^n \):

(i) \( 2c_2 + c_0 = 0 \) \( \Rightarrow \) \( c_2 = \frac{1}{2} \)

(ii) \( 6c_3 + 2c_1 + c_0 = 0 \) \( \Rightarrow \) \( c_3 = \frac{1}{6} \)

(iii) \( 12c_4 + 2c_2 + c_2 + c_1 = 0 \) \( \Rightarrow \) \( c_4 = -\frac{1}{8} \)

Therefore, the fourth degree polynomial approximation is

\[ y(x) \approx 1 + \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{8} \]
Express the polynomials $x^3$ and $x^3 + 2x^2 - 3x + 6$ in terms of Legendre polynomials.

**Solution:** Recall Rodrigue formula for Legendre polynomials

\[ P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n. \]

Clearly, $P_0(x) = 1$,

\[ P_1(x) = x, \quad \text{and one can easily find} \]

\[ P_2(x) = \frac{1}{2} (3x^2 - 1), \]

\[ P_3(x) = \frac{1}{2} (5x^3 - 3x). \]

Therefore, $x^3 = \frac{2}{5} P_3(x) + \frac{3}{5} P_1(x)$ and

\[ x^3 + 2x^2 - 3x + 6 = \frac{2}{5} P_3(x) + \frac{3}{5} P_1(x) + \left( \frac{4}{5} P_2(x) + \frac{1}{3} \right) \frac{1}{3}
\]

\[ -3 P_1(x) + 6 P_0(x). \]

\[ = \frac{2}{5} P_3(x) + \frac{4}{3} P_2(x) - \frac{12}{5} P_1(x) + \frac{20}{3} P_0(x) \]
Use first recurrence relation for Legendre polynomials to show that \( \frac{p_{500}^5}{p_{502}^5} \leq 0 \) if \( p_{501}^5 = 0 \).

**Solution:**

Recurrence relation for the Legendre polynomial is given by,

\[
(n+1) p_{n+1}(x) = (2n+1) x p_n(x) - np_{n-1}(x)
\]

Set \( x = L \) and \( n = 501 \), then the relation becomes

\[
1502 p_{502}(L) = (1003) L p_{501}(L) - (501) p_{500}(L)
\]

\[
1502 \frac{p_{502}(L)}{502} = (1003) L \frac{p_{501}(L)}{501} - (501) \frac{p_{500}(L)}{500}
\]

If \( p_{501} = 0 \), then \( \frac{p_{500}(L)}{500} = -\frac{502}{501} \leq 0. \)

Use first recurrence relation for Legendre polynomials to show that \( \int x p_n(x) p_m(x) \, dx = \frac{2n}{4n^2-1} \).

**Solution:**

First, we know that Legendre polynomial satisfies Orthogonality relation, that is

\[
\int_{-1}^{1} p_n(x) p_m(x) \, dx = \begin{cases} 0 & , \ m \neq n, \\ \frac{2}{2n+1} & , \ m = n. \\
\end{cases}
\]

By multiplying with \( p_n(x) \) in the recurrence relation and integrating over \(-1\) to \(+1\), we get,

\[
(n+1) \int_{-1}^{1} p_{n+1}(x) p_n(x) \, dx = (2n+1) \int_{-1}^{1} x p_n(x) p_n(x) \, dx - n \int_{-1}^{1} p_{n-1}(x) \, dx
\]

\[
\Rightarrow \int_{-1}^{1} x p_n(x) p_n(x) \, dx = \frac{2n}{(2n+1)(2n-1)} = \frac{2n}{4n^2-1}.
\]
Use first recurrence relation for Legendre polynomials to find the value of \( \int_{-1}^{1} x^2 p_n(x) \, dx \).

**Solution:** Rewrite the recurrence relation as

\[
(2n+1) x p_n(x) = (n+1) p_{n+1}(x) + np_{n-1}(x).
\]

\[
\Rightarrow (2n+1) x p_n(x) = (n+1) p_{n+1}(x) + 2n(n+1) p_{n+1}(x) p_{n-1}(x) + n^2 p_n(x).
\]

Now, integrate both sides from \(-1\) to \(1\) on both sides,

\[
(2n+1) \int_{-1}^{1} x^2 p_n(x) \, dx = (n+1)^2 \left( \frac{2}{2n+3} \right) + 0 + \frac{n^2}{2(n+1)}.
\]

\[
\int_{-1}^{1} x^2 p_n(x) \, dx = \frac{\frac{(n+1)^2}{(2n+3)} + \frac{n^2}{2(n+1)}}{2(n+1)^2 + 2n^2}.
\]

\[
\int_{-1}^{1} x^2 p_n(x) \, dx = \frac{\frac{(n+1)^2}{(2n+3)} + \frac{n^2}{(2n+1)(2n+3)}}{2n+1}.
\]