(1) Solve : $x(1-x^2)\,dy + (2x^2y - y - ax^3)\,dx = 0.$

Solution:

Rewrite the DE as ~~follows~~

$$\frac{dy}{dx} + \frac{(2x^2-1)}{x(1-x^2)}y = \frac{ax^2}{(1-x^2)}, \quad x \neq 0, \pm 1.$$

this is of the form $y' + p(x)y = Q(x)$ and it's general

Solution is $\quad y\,e^{\int p\,dx} = \int Q\,e^{\int p\,dx}\,dx + C \longrightarrow (i).$

where $p(x) = \dfrac{2x^2-1}{x(1-x^2)}$ & $Q(x) = \dfrac{ax^2}{(1-x^2)}.$

Now, Compute $\displaystyle\int p\,dx = -\int \frac{1-3x^2+x^2}{x(1-x^2)}\,dx$

$$= -\log\left(x(1-x^2)\right) + \frac{1}{2}\log(1-x^2)$$

$$= \log\left(\frac{1}{x\sqrt{1-x^2}}\right).$$

and $\displaystyle\int Q\,e^{\int p\,dx}\,dx = a\int \frac{x^2}{(1-x^2)}\,\frac{1}{x\sqrt{1-x^2}}\,dx$

$$= a\int d\left((1-x^2)^{-1/2}\right) = \frac{a}{\sqrt{1-x^2}}$$

Therefore, (i) implies,

$$\frac{y}{x\sqrt{1-x^2}} = \frac{a}{\sqrt{1-x^2}} + c \quad (\text{or}) \quad y = ax + cx\sqrt{1-x^2}$$

is the general Solution of the given DE.

(2) Solve $y \ln y \, dx + (x - \ln y) dy = 0$.

Solution: Rewrite the given DE as

$$\frac{dx}{dy} + \frac{1}{y \log y} x = \frac{1}{y}, \quad y \neq 0.$$

then, the general solution is

$$x e^{\int \frac{1}{y \log y} dy} = \int \frac{1}{y} e^{\int \frac{1}{y \log y} dy} dy + c, \quad y \neq 0$$

$$\Rightarrow \quad x e^{\log(\log y)} = \int \frac{1}{y} e^{\log(\log y)} dy + c, \quad y \neq 0$$

$$\Rightarrow \quad x \log y = \int \frac{\log y}{y} dy + c, \quad y \neq 0$$

$$\Rightarrow \quad x \log y = \frac{1}{2}(\log y)^2 + c, \quad y \neq 0.$$

~~is the general solution.~~

(3) Solve the IVP: $\frac{dy}{dx} = \frac{2}{x} y + x, \quad y(1) = 2.$

Solution: the general solution for given DE is

$$y e^{-\int \frac{2}{x} dx} = \int Q e^{\int -\frac{2}{x} dx} dx + c, \quad c - \text{arbitrary Constant.}$$

$$\Rightarrow \frac{y}{x^2} = \log x + c$$

by initial Condition, $\boxed{c = 2}$ & hence, $\boxed{y = x^2 \log x + 2x^2}$

is the general solution for the DE

(4) Solve: $\frac{dy}{dx} + \frac{y}{x}\log y = \frac{y}{x^2}(\log y)^2$.

Solution: Rewrite the given DE as

$$\frac{1}{y(\log y)^2}\frac{dy}{dx} + \frac{1}{x\log y} = \frac{1}{x^2}, \quad y\neq 0,\ x\neq 0. \longrightarrow (i).$$

let $u = (\log y)^{-1}$ then, $\frac{du}{dx} = \frac{1}{(\log y)^2}\left(\frac{-1}{y}\right)\frac{dy}{dx}$.

Now, (i) $\Rightarrow$ $-\frac{du}{dx} + \frac{1}{x}u = \frac{1}{x^2} \longrightarrow$ (ii)

the general solution for (ii) is

$$u\, e^{-\int\frac{1}{x}dx} = \int \frac{-1}{x^2} e^{-\int\frac{1}{x}dx}\,dx + c, \quad c - \text{constant}.$$

$$\Rightarrow \frac{u}{x} = \frac{+1}{2x^2} + c$$

(or) $\frac{2x}{\log y} = 1 + 2x^2 c$ (or) $\log y = \frac{2x}{1+2cx^2}$

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(5) Solve: $x^3\frac{dy}{dx} - x^2 y + y^4\cos x = 0$.

Solution: Rewrite given DE as,

$$\frac{1}{y^4}\frac{dy}{dx} - \frac{1}{y^3}\frac{1}{x} = -\frac{\cos x}{x^3} \longrightarrow (i).$$

Let $u = y^{-3}$, then $\frac{du}{dx} = \frac{-3}{y^4}\frac{dy}{dx}$