Assignment – II

① Solve the boundary value problem associated ~~with~~ to heat distribution in a finite bar with insulated ends as given by,

$$u_t - c^2 u_{xx} = 0, \quad 0 < x < L, \ t > 0,$$

$$u(x,0) = e^{-x}, \quad 0 \le x \le L,$$

$$u_x(0,t) = 0 = u_x(L,t), \quad t > 0.$$

Solution:

Let $u(x,t) = X(x)T(t) \neq 0$. then by given heat eqn,

$$\frac{T'(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} \ (= \lambda, \text{ say a constant})$$

$$\Rightarrow \quad X''(x) - \lambda X(x) = 0, \quad 0 < x < L$$

& by boundary Conditions, $X'(0) = 0$ & $X'(L) = 0$. Note that this system is regular S-L problem, Hence, "λ – real"

For $\lambda = 0$:

$$\left.\begin{aligned} &X(x) = C_1 x + C_2 \\ &X'(0) = 0 \Rightarrow C_1 = 0 \\ &X'(L) = 0 \Rightarrow 0 = 0 \text{ satisfied} \Rightarrow X(x) = C_2 \end{aligned}\right\} \Rightarrow$$

$\lambda = 0$ is an eigenvalue with eigenvect $X(x) = 1$ (const

For $\lambda = \mu^2$:

$$X(x) = C_1 e^{\mu x} + C_2 e^{-\mu x}$$

$$X'(x) = \mu(C_1 e^{\mu x} - C_2 e^{-\mu x})$$

$$X'(0) = 0 \Rightarrow C_1 - C_2 = 0 \Rightarrow X(x) = 2C_1 \cosh \mu x.$$

$$X'(L) = 0 \Rightarrow 2\mu C_1 \sinh \mu L = 0 \Rightarrow \cancel{\text{satisfied}} \ C_1 = 0$$

$$\Rightarrow X(x) = 0, \quad \lambda = \mu^2 \text{ is not an eigenvalue}$$

① contin...

For $\lambda = -\mu^2$ $(\mu \neq 0)$

$$X(x) = C_1 \cos \mu x + C_2 \sin \mu x$$

$$X'(0) = 0 \Rightarrow C_2 = 0$$

$$X'(L) = 0 \Rightarrow C_1 \mu \sin \mu L = 0 \Rightarrow \mu L = n\pi, \ n = 0, 1, ..$$

therefore, Eigenvalues are $\lambda_n = -\left(\frac{n\pi}{L}\right)^2$, $n = 0, 1, 2, \dots$

Eigenfunctions are $X_n(x) = \cos \frac{n\pi}{L} x$, $n = 0, 1, 2, \dots$

Now, for these eigenvalues λ_n, the general solution for

$$T_n'(t) + \left(\frac{n\pi}{L}\right)^2 c^2 T_n(t) = 0 \quad t > 0, \quad n = 0, 1, 2, \dots$$

is $T_n(t) = A_n e^{-\left(\frac{n\pi c}{L}\right)^2 t}$, $n = 0, 1, 2, ..$

Let $u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = \sum_{n=0}^{\infty} A_n e^{-\left(\frac{n\pi c}{L}\right)^2 t} \cos\left(\frac{n\pi}{L} x\right)$ be the solution of the heat equation that satisfies the given BC's.

By initial condition $u(x,0) = e^{-x}$, $0 < x < L$

$$\Rightarrow \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi}{L} x\right) = e^{-x}, \quad 0 < x < L$$

$$\Rightarrow A_n = \int_0^L e^{-x} \cos\left(\frac{n\pi x}{2}\right) dx \Big/ \int_0^L \cos^2\left(\frac{n\pi}{L} x\right) dx, \quad n = 1, 2..$$

$$= \left(\frac{L}{2}\right)^{-1} \cdot \frac{L^2}{L^2 + (n\pi)^2} \left[-e^{-L} \cos n\pi + 1 \right] = \frac{2L}{L^2 + n^2\pi^2} \left(1 - (-1)^n e^{-L}\right).$$

$$A_0 = \frac{1}{L}\left(1 - e^{-L}\right).$$

Hence, the required solution is,

$$u(x,t) = \frac{1}{L}\left(1 - e^{-L}\right) + \sum_{n=1}^{\infty} \frac{2L\left(1 - (-1)^n e^{-L}\right)}{L^2 + n^2\pi^2} \cos\left(\frac{n\pi}{L} x\right) e^{-\left(\frac{n\pi c}{L}\right)^2 t}, \quad t \geq 0, \ 0 \leq x \leq L.$$