Assignment - 10

① Find the solution of the initial boundary value problem

$$u_{tt} - c^2 u_{xx} = 0 \;, 0 < x < L \;, t \geq 0.$$

$$\text{IC:} \begin{cases} u(x,0) = \sin^3\left(\frac{\pi x}{L}\right), & 0 \leq x \leq L, \\ u_t(x,0) = 0, & 0 \leq x \leq L, \end{cases}$$

$$\text{BC:} \begin{cases} u(0,t) = 0, \\ u(L,t) = 0 \end{cases}$$

Solution: Let $u(x,t) = X(x)T(t) \; (\neq 0)$, then

$$X(x)T''(t) - c^2 X''(x)T(t) = 0$$

$$\Rightarrow \frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = \lambda \text{ (say, some constant } \lambda)$$

$$\Rightarrow X''(x) - \lambda X(x) = 0 \;\; \& \;\; T''(t) - \lambda c^2 T(t) = 0$$

The boundary conditions becomes $X(0) = 0 \;\&\; X(L) = 0$

Note here that, $\begin{cases} X''(x) - \lambda X(x) = 0, 0 < x < L, \\ X(0) = 0, \\ X(L) = 0, \end{cases}$ is a regular

Strum-Liouvillie system and λ is real constant.

Now, it is easy to see that (write details in exams)

$\left\{X_n(x) = \sin\left(\frac{n\pi}{L}x\right), n = 1, 2, \dots\right\}$ are the eigen-function corresponding

to the eigen-values $\left\{\lambda_n = -\left(\frac{\pi n}{L}\right)^2; \; n = 1, 2, \dots\right\}$.

① - continue.

Now for $\lambda_n = -\left(\frac{n\pi}{L}\right)^2$, the general solution for $T_n''(t) + \left(\frac{cn\pi}{L}\right)^2 T_n(t) = 0$ is ~~given~~ obtained as

$$T_n(t) = A_n \cos\left(\frac{cn\pi t}{L}\right) + B_2 \sin\left(\frac{cn\pi}{L} t\right), \quad t>0$$

Therefore, $u_n(x,t) = \left(A_n \cos\left(\frac{cn\pi}{L} t\right) + B_n \sin\left(\frac{cn\pi}{L} t\right)\right) \sin\left(\frac{n\pi x}{L}\right)$, $0<x<L$, $t \geq 0$.

and by Super-position principle,

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$= \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{cn\pi}{L} t\right) + B_n \sin\left(\frac{cn\pi}{L} t\right)\right) \sin\left(\frac{n\pi x}{L}\right).$$

Now; Apply initial Conditions.

(i) $u(x,0) = \sin^3\left(\frac{\pi x}{L}\right)$, $0 \leq x \leq L$ implies

$$\sin^3\left(\frac{\pi x}{L}\right) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right).$$

$$\Rightarrow A_n = \frac{\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin^3\left(\frac{\pi x}{L}\right) dx}{\int_0^L \sin^2\left(\frac{n\pi}{L} x\right) dx}$$

$$\Rightarrow A_n = \frac{2}{L} \int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin^3\left(\frac{\pi x}{L}\right) dx$$

① continuous

Instead of computing the integrals in A_n, look at the series,

$$\sin^3\left(\frac{\pi}{L}x\right) = A_1 \sin\left(\frac{\pi x}{L}\right) + A_2 \sin\left(\frac{2\pi x}{L}\right) + A_3 \sin\frac{3\pi}{L}x + \cdots$$

and the LHS is $\sin^3\left(\frac{\pi}{L}x\right) = \frac{1}{4}\left[3\sin\left(\frac{\pi}{L}x\right) - \sin\left(\frac{3\pi}{L}x\right)\right].$

therefore $A_1 = \frac{3}{4}$, $A_2 = 0$, $A_3 = \frac{-1}{4}$ & $A_n = 0$ $\forall n \geq 4$.

The ~~boundary~~ initial condition (ii) $u_t(x,0) = 0$ implies,

$$0 = \sum_{n=1}^{\infty} \left(\frac{cn\pi}{L}\right) B_n \sin\left(\frac{n\pi x}{L}\right), \quad 0 \leq x \leq L.$$

$$\Rightarrow \quad B_n = 0 \quad \forall n \geq 1.$$

Therefore, the required solution is

$$u(x,t) = \frac{3}{4}\cos\left(\frac{c\pi}{L}t\right)\sin\frac{\pi}{L}x - \frac{1}{4}\cos\left(\frac{3c\pi}{L}t\right)\sin\left(\frac{3\pi}{L}x\right),$$

$$0 \leq x \leq L, \quad t \geq 0$$

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② Find the solution of the initial value problem

$$u_{tt} - c^2 u_{xx} = 0, \quad 0 < x < L, \ t > 0$$

$$u(x,0) = \begin{cases} x, & 0 \leq x \leq L/2 \\ L - x, & \frac{L}{2} < x < L. \end{cases}$$

$$u_t(x,0) = 0, \quad 0 \leq x \leq L.$$

$$u(0,t) = 0 = u(L,t), \quad t \geq 0$$

Solution: