# Assignment - 1

(1). Solve $\sqrt{1+x^2+y^2+x^2y^2}\,dx + xy\,dy = 0$

**Solution:** rewrite the differential eqn as

$$\sqrt{(1+x^2)(1+y^2)}\,dx + xy\,dy = 0$$

$$\Rightarrow \frac{\sqrt{1+x^2}}{x}\,dx + \frac{y}{\sqrt{1+y^2}}\,dy = 0, \quad x \neq 0.$$

$$\Rightarrow \int \frac{\sqrt{1+x^2}}{x}\,dx + \int \frac{y\,dy}{\sqrt{1+y^2}} = c \quad \text{(by Integration)} \longrightarrow (i)$$

Clearly, $\displaystyle\int \frac{y\,dy}{\sqrt{1+y^2}} = \int d\left(\sqrt{1+y^2}\right) = \sqrt{1+y^2}$.

To evaluate $\displaystyle\int \frac{\sqrt{1+x^2}}{x}\,dx$, set $t^2 = 1+x^2$

$\Rightarrow t\,dt = x\,dx$

$\Rightarrow t\,dt = (t^2-1)\dfrac{dx}{x}$

$$\int \frac{\sqrt{1+x^2}}{x}\,dx = \int \frac{t^2}{t^2-1}\,dt$$

$$= \int \frac{t^2-1}{t^2-1}\,dt + \int \frac{1}{t^2-1}\,dt$$

$$= t + \frac{1}{2}\log\left|\frac{t-1}{t+1}\right|$$

$$= \sqrt{1+x^2} + \frac{1}{2}\log\left[\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right].$$

Therefore, the solution is (from (i))

$$\sqrt{1+x^2} + \sqrt{1+y^2} + \frac{1}{2}\log\left[\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right] = c, \quad x \neq 0$$

$c$ – arbitrary constant.

Q Solve : $\dfrac{dy}{dx} = \sin(x+2y) + \cos(x+2y)$.

Solution: Let $t = x+2y$ then $\dfrac{dt}{dx} = 1 + 2\dfrac{dy}{dx}$ and the differential equation becomes,

$$\frac{dt}{dx} - 1 = 2\sin t + 2\cos t \qquad \text{(or)}$$

$$\frac{1}{1+2\sin t+2\cos t}\,dt - dx = 0 .$$

arbitrary constant.

Clearly, the solution is : $\displaystyle\int \frac{dt}{1+2\sin t+2\cos t} = \int dx + C \longrightarrow \text{(i)}$

In order to integrate LHS of (i), rewrite: $\sin t = \dfrac{2\tan t/2}{1+\tan^2 t/2}$

and $\cos t = \dfrac{1-\tan^2 t/2}{1+\tan^2(t/2)}$,

then, $\displaystyle\int \frac{dt}{1+2\sin t+2\cos t} = \int \frac{(1+\tan^2 t/2)\,dt}{1+\tan^2 t/2 + 4\tan t/2 + 2 - 2\tan^2 t/2}$

Set $s = \tan t/2$

$\Rightarrow \dfrac{ds}{dt} = \dfrac{1}{2}\sec^2 t/2$

(or) $ds = \dfrac{1}{2}(1+\tan^2 t/2)\,dt$

$$= \int \frac{2\,ds}{3 - s^2 + 4s}, \text{ where } s = \tan t/2.$$

$$= -2\int \frac{ds}{(s-2)^2 - 7}$$

$$= \frac{-1}{\sqrt{7}} \int \left(\frac{1}{(s-2)-\sqrt{7}} - \frac{1}{(s-2)+\sqrt{7}}\right) ds$$

$$= \frac{-1}{\sqrt{7}} \log\left|\frac{s-2-\sqrt{7}}{s-2+\sqrt{7}}\right|.$$

Therefore, the solution (i) becomes,

$$\frac{-1}{\sqrt{7}} \log\left|\frac{\tan\left(\frac{x+2y}{2}\right) - 2 - \sqrt{7}}{\tan\left(\frac{x+2y}{2}\right) - 2 + \sqrt{7}}\right| = x + C$$

(3) Solve : $(x^3+y^3)\,dx - (x^2y+xy^2)\,dy = 0$.

Solution: Write the given diff. equation as

$$\frac{dy}{dx} = \frac{x^3+y^3}{x^2y+y^2x} \quad \left(\text{observe: equation is homogeneous of degree } 0\right).$$

Set $y = vx$ ; $\frac{dy}{dx} = v + x\frac{dv}{dx}$, then, the differential equation becomes,

$$v + x\frac{dv}{dx} = \frac{1+v^3}{v+v^2}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1+v^3-v^2-v^3}{v+v^2}$$

$$\Rightarrow \quad \int \frac{v}{1-v}\,dv = \int \frac{dx}{x} + C \quad , \quad C\text{- arbitrary Constant.}$$

$$\Rightarrow \quad -v - \log|1-v| = \log|x| + C$$

$$\Rightarrow \quad -\frac{y}{x} - \log|1-y/x| = \log|x| + C \quad , \quad x \neq 0.$$

$$\Rightarrow \quad -y/x - \log|x-y| = C \quad , \quad x \neq 0.$$

(or)

$$\frac{y}{x} + \log(x-y) + C = 0 \quad , \quad x \neq y, \; x \neq 0,$$

is the general solution for the given differential equation.