(1) Let $A$ be a subring of $B$ such that $B$ is integral over $A$. Let $n$ be a maximal ideal of $A$. Is $B_n$ necessarily integral over $A_m$, where $m = n \cap A$?

**Solution:** Let $A = k[x^2 - 1]$ and $B = k[x]$. Then $B$ is integral over $A$. Note that $n = (x - 1)$ is a maximal ideal in $B$ and $m = (x^2 - 1)$ is a maximal ideal in $A$ with $m = n \cap A$. Suppose $\frac{1}{x+1} \in B_n$ is integral over $A_m$. Then there exists $n > 0$, $\frac{a_n}{s_n} \in A_m$ such that

$$\frac{1}{(x+1)^n} + \frac{a_{n-1}}{s_{n-1}} \frac{1}{(x+1)^{n-1}} + \cdots + \frac{a_0}{s_0} = 0.$$ 

After clearing the denominators we get

$$z \sum_{i=0}^{n} a_i t_i (x+1)^{n-i} = 0,$$

where $a_n = 1$, $t_i = s_{n-1} \cdots s_{i-1} s_{i+1} \cdots s_0$, $0 \leq i \leq n-1$, $t_n = s_{n-1} \cdots s_0$ and $z \in B \setminus n$. We have

$$za_0 t_0 (x+1)^n + \cdots + z t_n = 0$$

which implies that

$$(x+1) \{za_0 t_0 (x+1)^{n-1} + \cdots + za_{n-1} t_{n-1}\} = -zt_n.$$ 

Therefore $x+1 | zt_n$. Since $x+1$ is a prime element, $x+1 | z$, $z \in (x+1)B \cap A = (x^2 - 1)$. This is contraction to $f \in B \setminus n$. Therefore $\frac{1}{(x+1)}$ is not integral over $A_m$.

(2) Let $B$ be an integral extension of $A$. Prove that

(a) If $x \in A$ is a unit in $B$, then it is a unit in $A$.

**Solution.** Since $B$ is integral over $A$, $1/x$ satisfies

$$\frac{1}{x^n} + a_1 \frac{1}{x^{n-1}} + \cdots + a_n = 0,$$

where $a_i \in A$, for $1 \leq i \leq n$. Now multiply by $x^{n-1}$, we get

$$\frac{1}{x} + a_1 + \cdots + a_{n-1} x^{n-2} + a_n x(n-1) = 0,$$

and

$$\frac{1}{x} = -(a_1 + \cdots + a_{n-1} x^{n-2} + a_n x(n-1)) \in A.$$ 

Therefore $x$ is a unit in $A$.

(b) the Jacobson radical of $A$ is the contraction of the Jacobson radical of $B$.

**Solution.** Let $J_A$ and $J_B$ be Jacobson radical of $A$ and $B$ respectively. We need to prove that $J_A = J_B \cap A$. Let $x \in J_B \cap A$. Then for any $a \in A$, $1 + ax \in A$ is a unit in $B$. By (a), $1 + ax$ is a unit in $A$. Therefore $x \in J_A$. 


Let \( x \in J_A \). For a maximal ideal \( \mathfrak{m}_B \) of \( B \), \( \mathfrak{n} = \mathfrak{m}_B \cap A \) is a maximal ideal in \( A \).

Therefore \( x \in \mathfrak{n} = \mathfrak{m}_B \cap A \subseteq \mathfrak{m}_B \). Hence \( x \in J_B \cap A \).

(3) If \( A \) is an integral domain, then prove that \( A \) is integrally closed in \( A[x] \). Give an example of \( A \) such that \( A \) is not integrally closed in \( A[x] \).

**Solution.** Suppose \( f(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0 \in A[x] \setminus A \), \( b_m \neq 0 \), is integral over \( A \). Then there is a monic polynomial \( g(t) = a_0 + a_1 t + \cdots + a_{n-1} t^{n-1} + t^n \in A[t] \) such that \( g(f(x)) = 0 \). Therefore

\[
b_m^n x^{mn} + \cdots + (a_n + a_{n-1} b_0^n - \cdots + b_0^n) = 0.
\]

Hence \( b_m^n = 0 \) so that \( b_m = 0 \). This is a contradiction to the assumption that \( b_m \neq 0 \). Therefore \( A \) is integrally closed in \( A[x] \).

Consider \( A = \mathbb{Z}_4 \) and \( f(x) = 2x + 2 \in \mathbb{Z}_4[x] \). Note that \( f(x)^2 = 0 \). Therefore \( f(x) \) is integral over \( A \) but \( f(x) \notin A \).