(1) Suppose that for each prime ideal \( p \subset A \), the local ring \( A_p \) has no nonzero nilpotent elements. Prove that \( A \) has no nonzero nilpotent elements.

**Solution:** Suppose \( x \) is a nonzero nilpotent element. Note that, if \( x \) is nilpotent in \( A \), then \( x/1 \) is also nilpotent in \( A_q \). Therefore \( \frac{x}{1} = 0 \in A_q \) for all prime ideals \( q \). Since \( x \) is nonzero nilpotent, \( (0 : x) \) is a proper ideal. Let \( p \) be a prime containing \( (0 : x) \). Since \( \frac{x}{1} = 0 \in A_p \), there exists \( s \in A \setminus p \) such that \( sx = 0 \), i.e., \( s \in (0 : x) \subset p \) which is a contradiction. Hence \( x = 0 \).

(2) Let \( I \) be an ideal and let \( S = 1 + I = \{ 1 + x : x \in I \} \). Prove that \( S \) is a multiplicatively closed subset. Prove that \( S^{-1}I \) is contained in the Jacobson radical of \( S^{-1}A \).

**Solution:** Note that \( 0 \notin S \) and \( 1 \in S \). If \( 1 + x, 1 + y \in S \), then \( (1 + x)(1 + y) = 1 + (x + y + xy) \in S \). Therefore \( S \) is a multiplicatively closed subset.

Let \( x/s \in S^{-1}I \), where \( s \in S \) and \( x \in I \). We need to show that \( x/s \) is in the Jacobson radical of \( S^{-1}A \). For \( r/t \in S^{-1}A \), where \( t \in S \) and \( r \in A \),

\[
1 + \frac{rx}{ts + rx} = \frac{ts}{ts}.
\]

Since \( t \) and \( s \) are in \( S = 1 + I \) and \( rs \in I \), we see that \( ts \in S \) and \( ts + rx \in S \). Therefore we conclude that \( (ts + rx)/(ts) \) is a unit in \( S^{-1}A \). Hence \( \frac{x}{s} \) is in the Jacobson radical of \( S^{-1}A \).

(3) For two ideals \( I, J \) in \( A \), prove that \( I \subset J \) if and only if \( I_m \subset J_m \) in \( A_m \) for all maximal ideal \( m \).

**Solution:** If \( I \subset J \), then \( S^{-1}I \subset S^{-1}J \) in \( S^{-1}A \) for any multiplicative set \( S \).

Conversely, let \( I_m \subset J_m \) in \( A_m \) for all maximal ideals \( m \). Suppose \( I \nsubset J \). Then \( \frac{I + J}{J} \neq 0 \) i.e., there exists an element \( 0 \neq z \in \frac{I + J}{J} \). Therefore, \( (0 : z) \) is a proper ideal in \( A \) and hence contained in a maximal ideal, say \( m \). We have

\[
\left( \frac{I + J}{J} \right)_m \approx \frac{I_m + J_m}{J_m} = 0.
\]

Therefore \( z/1 = 0 \). Then there exist a \( t \in A \setminus m \) such that \( tz = 0 \). Hence \( t \in (0 : z) \). This contradicts the assumption that \( (0 : z) \cap (A \setminus m) = \emptyset \). Therefore \( I \subset J \).

(4) Is \( \sqrt{2} + \sqrt{3} + \frac{3}{2} \sqrt{3} \in \mathbb{R} \) integral over \( \mathbb{Z} \)? Justify your answer.

**Solution:** Let \( C = \{ x \in \mathbb{R} \mid x \text{ is integral over } \mathbb{Z} \} \). Then \( -\sqrt{2} + \sqrt{2} \in C \). Therefore, \( \sqrt{2} + \sqrt{2} + \frac{3}{2} \sqrt{3} \in C \), then \( \frac{1}{2} \sqrt{3} \in C \) and hence \( \frac{3}{2} \in C \) which is a contradiction since \( \mathbb{Z} \) is integrally closed in \( \mathbb{Q} \). Therefore \( \sqrt{2} + \sqrt{2} + \frac{3}{2} \sqrt{3} \) is not integral over \( \mathbb{Z} \).