Quiz 1

1. Which of the following statements are true.

   (a) Every even integer is divisible by 4 if and only if either 7 divides 21 or 9 divides 12.

   (b) Either snow is hot or 2 is even implies 3 is even

   (c) $\forall x \in \mathbb{Z}$, $3x^2 + 2x + 3 \neg 0$ implies $\exists x \in \mathbb{Z}$ such that $3x^2 + 2x + 3 = 0$.

   (d) $\forall$ even $x, y \in \mathbb{N}$ $\exists k \in \mathbb{N}$, $x^2 + y^2 = 4k$ or $x^2 + y^2 = 4k + 1$.

   Ans : d . If x and y are both even then $x = 2s$ and $y = 2r$ for some integer r, s respectively. Then $x^2 + y^2 = 4r^2 + 4s^2 = 4(r^2 + s^2)$ and hence is of the form $4k$.

2. Write the following expression are equal to $(p \implies q)$

   (a) $q \implies p$

   (b) $\neg p \implies \neg q$

   (c) $[\neg p \wedge (p \vee q)] \implies q$

   (d) $\neg q \implies \neg p$

   Ans : d. Straightforward to check. You can check like this case by case. $(p \implies q)$ means if p is true 1 has to be true i.e. if $p = 1$ then $q = 1$. If $p = 0$ then 1 can e 0 or 1. Check which case is satisfied.

3. Write the following expression are equal to $(p \implies q)$

   (a) $(p \vee \neg q)$

   (b) $(\neg p \vee q)$

   (c) $(p \wedge q)$

   (d) $(p \wedge q) \vee \neg p$

   Ans : b & d . Check the same thing as aforementioned.

4. Write the following expression are equal to $(p \iff q)$

   (a) $q \iff p$

   (b) $\neg p \implies \neg q$

   (c) $\neg q \implies \neg p$

   (d) $(\neg q \implies \neg p) \wedge (\neg p \implies \neg q)$

Ans : a & d. ($p \iff q$) means either both p, q are false(i.e. 0) or both are true.

5. Write the following expression are equal to ($p \iff q$)

   (a) $(p \vee \neg q)$

   (b) $(\neg p \vee q)$

   (c) $(p \wedge q)$

   (d) $(p \wedge q) \vee (\neg p \wedge \neg q)$.

   Ans : d . Same logic.

6. If $|A^c| = 18$ and $|B^c| = 24$ and $|(A \cup B)^c| = 12$ and $|A \cap B| = 3$ what is the $|A \cup B|$?

   Ans : 21. Use vein-diagram. Suppose the A part where it is without the intersection is x(i.e. $|A - A \cap B| = x$), intersection is y (i.e. $|A \cap B| = y$), B part without intersection is z (i.e. $|B - A \cap B| = z$) and the whole part outside union of A and B is w (i.e. $|A \cup B|^c = w$). Then we have basically $z + w = 18, x + w = 24, w = 12, y = 3$. Then we have $z = 6$ and $x = 12$. Now $|A \cup B| = x + y + z = 21$

7. How many functions are there from $\{-1, 0, 1\}^3$ to $\{-1, +1\}$?

   Ans : $2^{27} = 134217728$. In general suppose we have $S = \{f | f : X \to Y\}$. Then $|S|$ i.e. # all possible functions from X to Y is precisely $|Y|^{|X|}$. Here we have $|Y| = 2$ and $|Z| = 3^3 = 27$.

8. If A, B and C are three sets such that $|A| = |B| = |C| = 18$ and $|A \cup B \cup C| = 36$ and $|A \cap B| = |A \cap C| = |B \cap C| = 7$ then what is the size of $|A \cap B \cap C|$.

   Ans; 3. From inclusion-exclusion we have $36 = |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| = 3 \times 18 - 3 \times 7 + |A \cap B \cap C| = 33 + |A \cap B \cap C| \implies |A \cap B \cap C| = 3$

9. If n is a positive odd integer. Then $n^2 \equiv x (\text{mod } 16)$. Which of the following is/are possible value(s) of x.

   (a) 1

   (b) 5

   (c) 9

   (d) 13

   Ans : a & c . Trivial to check . Actually check that $1^2, 3^2, 5^2, 7^2 \equiv 1$ or 9 mod 16 . Now all the odd numbers can be of the form $8k + r$ where $r = 1, 3, 5, 7$. Hence $(8k + r)^2$ mod $16 = 64k^2 + 16kr + r^2$ mod $16 \equiv r^2$ mod 16.

10. if p is a prime then p can be congruent to which of the following modulo 15.

    (a) 2

    (b) 5

    (c) 7

(d) 9

Ans : a & b & c . 2,5,7 are itself primes that satisfies. p can not be $\equiv 9 \bmod 15$ because that that means $p = 15k + 9$ for some $k \geq 0$. Hence $p = 3(5k + 3)$ i.e. 3 always divides p and $5k + 3 > 1$ and so can not be prime.

11. Let x be an integer. When can $x^2 - 13x + 5$ be even.

   (a) When x is odd
   (b) When x is even
   (c) always
   (d) Never

   Ans : d. $x^2 - 13x = x(x - 13)$. Now for any integer x , x and $x - 13$ has different parity i.e. if x is odd $x - 13$ is even and if x is even then $x - 13$ is odd. But that means for any x $x(x - 13)$ is even. Hence $x^2 - 13x + 4$ is always even and hence $x^2 - 13x + 5$ is always odd.

12. The following statement

$$\forall a_1, a_2, a_3, a_4 \in \mathbb{R} \quad \frac{a_1 + a_2 + a_3 + a_4}{4} \geq \sqrt[4]{a_1 a_2 a_3 a_4}$$

   (a) is always true
   (b) is true only when all a_is are positive
   (c) is true only when exactly two or four of the a_is are positive
   (d) True only when a_is are equal.
   (e) Never true.

   Ans : b. When all are positive then using question 20, we have $a_1 + a_2 \geq 2\sqrt{a_1 a_2}$ and $a_3 + a_4 \geq 2\sqrt{a_3 a_4}$. So $a_1 + a_2 + a_3 + a_4 \geq 2\sqrt{a_1 a_2} + 2\sqrt{a_3 a_4} = 2(\sqrt{a_1 a_2} + \sqrt{a_3 a_4}) \geq 4\sqrt[4]{a_1 a_2 a_3 a_4}$. It might happen that 3 of them are positive and other one is negative . Then RHS does not make sense as under the root that quantity is negative.

13. If x, y, z are two real numbers then which of the following is/are true

   (a) $x^2 + y^2 + z^2 \geq 2xy$
   (b) $x^2 + y^2 + z^2 \geq xy + xz + yz$
   (c) $x^2 + y^2 + z^2 \geq 2xy + 2yz$
   (d) None of the above

   Ans: a & b. $x^2 + y^2 + z^2 - 2xy = (x - y)^2 + z^2 \geq 0$ and $x^2 + y^2 + z^2 - (xy + xz + yz) = \frac{1}{2}\{(x - y)^2 + (y - z)^2 + (z - x)^2\} \geq 0$. c is not true because for example take $x = y = z = 1$ then l.h.s=3 where r.h.s=4.

14. If a and b are two distinct odd primes then which of the following is/are true.

(a) $a^2 + b^2 \geq 36$

(b) $a^2 + b^2 \geq 34$ or $a + b \leq 8$.

(c) $(a + b)^2 \geq 4ab$

(d) None of the above.

Ans : b & c. If $a + b \leq 8$ then it has to be $a = 3, b = 5$ or vice-versa. If not then already we have $3^2 + 5^2 = 34$ so $a^2 + b^2 \geq 34$ will be true. c is anyhow satisfied for any a and b because $(a + b)^2 - 4ab = (a - b)^2 \geq 0$.

15. Which of the following is true

(a) Sum of two rational numbers is an rational number.

(b) Product of two rational numbers in an rational number

(c) Square root of an rational number is an rational number

(d) Square of an rational number is an rational number.

Ans; a &b &d . $\frac{p}{q} + \frac{r}{s} = \frac{ps+rq}{sq}$, $\frac{p}{q} \times \frac{r}{s} = \frac{pr}{qs}$, $(\frac{p}{q})^2 = \frac{p^2}{q^2}$. c is not true as for example take 2. $\sqrt{2}$ is not rational.

16. A number that is not a rational is called an irrational number. Which of the following is true

(a) Sum of two irrational numbers is an irrational number.

(b) Product of two irrational numbers in an irrational number

(c) Square root of an irrational number is an irrational number

(d) Square of an irrational number is an irrational number.

Ans : c. sum,product and square of irrational numbers can be rational. For example $\sqrt{2}$ is irrational. But take $\sqrt{2}$ and $-\sqrt{2}$. Sum=0 ,product $=-2$. $(\sqrt{2})^2 = 2$. c is true because suppose $\sqrt{x} = \frac{p}{q}$ where x is irrational. Then $x = \frac{p^2}{q^2}$ and hence x is itself a rational number a contradiction.

17. If k and ℓ are two positive integers then which of the following is possible.

(a) $k^2 - \ell^2 = 2$

(b) $k^2 - \ell^2 = 4$

(c) $k^2 - \ell^2 = 5$

(d) $k^2 - \ell^2 = 102$

Ans : c. $3^2 - 2^2 = 5$. If $k^2 - l^2$ is even then it has to be divisible by 4 because $k^2 - l^2 = (k + l)(k - l)$ and if that is even then one of them has to be even . But $k + l = (k - l) + 2l$ and so parity of $k + l$ and $k - l$ are same and infact in this case both are even and hence is divisible by 4. Hence a and d is ruled out. b is ruled out because $(k + l)(k - l) = 4$ hence $k + l = 2, -2$ ($k + l$ can not be 4 or -4 because then $k - l$ would be 1 or -1 which is not possible by above observation.) But then wither k or l has to be 0 which is not possible.

18. For any $n \in \mathbb{Z}^+$ which of the following is true

   (a) $\sqrt{n} + \sqrt{2}$ is not rational.

   (b) $\sqrt[3]{n} + \sqrt{2}$ is not rational.

   (c) $\sqrt{2+n}$ is not rational.

   (d) $\sqrt{2+4n}$ is not rational.

   Ans : a & b & d. Assume for a,b, or d that that is $= \frac{p}{q}$ and show a contradiction. For first one just square it and get contradiction. For (b) raise it to the power 6. For (d) 4n+2 can never be a square of an integer and hence the contradiction. (c) might be true as take n=14 then $\sqrt{2+n} = 4$ an integer (i.e rational) .

19. If a is a positive integer, then $a^2 + a^4 \equiv 0 (\text{mod } 5)$ if

   (a) $a \equiv 2 (\text{mod } 5)$

   (b) $a \equiv 3 (\text{mod } 5)$

   (c) 5 divides a.

   (d) None of the above.

   Ans : a & b & c. If $a \equiv 2 \bmod 5$ then $a^4 = 2^4 \bmod 5 = 1 \bmod 5$ and $a^2 \equiv 4 \bmod 5$ hence $a^4 + a^2 \equiv 0 \bmod 5$. Likewise if $a \equiv 3 \bmod 5$ then $a^4 = 1 \bmod 5$ and $a^2 \equiv 4 \bmod 5$ and so $a^2 + a^4 \equiv 0 \bmod 5$. If $a \equiv 0 \bmod 5$ then trivially $a^4, a^2 \equiv 0 \bmod 5$ and so their sum.

20. If for any real numbers a and b $\frac{a+b}{2} \geq \sqrt{ab}$ then This statement is

   (a) Always true

   (b) True only when a and b are positive

   (c) True only when $a = b$.

   (d) Never true.

   Ans : b. For positive it is true as $\frac{a+b}{2} - \sqrt{ab} = \frac{1}{2}(\sqrt{a} - \sqrt{b})^2 \geq 0$. See for negative a,b it might be the case that a is positive and b is negative then RHS does not make sense.