Assignment 5: solutions

1. If \( f(x, y) = (-y, x) \) and \( g(x, y) = (\frac{x-y}{\sqrt{2}}, \frac{x+y}{\sqrt{2}}) \), then the composition of \( g \) with itself produces \( f \), i.e., \( g \circ g = f \). \( \text{True} \)

2. Let \( f \) be an arbitrary affine transformation of \( \mathbb{R}^2 \). Let \( f(1, 1) = (p, q) \). Then \( f(2, 2) = (2p, 2q) \). \( \text{False} \)

   Take \( f \) to be a translation; say \( f(x, y) = (x + 1, y + 1) \). This does not satisfy the given relation.

3. Let \( f \) be an arbitrary linear transformation of \( \mathbb{R}^2 \). Let \( f(1, 1) = (p, q) \). Then \( f(2, 2) = (2p, 2q) \). \( \text{True} \)

4. Let \( f \) be an arbitrary linear transformation of \( \mathbb{R}^2 \). The image of the unit circle \( x^2 + y^2 = 1 \) under \( f \) is a circle. \( \text{False} \)

   It could also be an ellipse.

5. There is a unique linear transformation of \( \mathbb{R}^2 \) which maps the \( X \)-axis to the line \( y = 2x \), and the \( Y \)-axis to the line \( y = x \). \( \text{False} \)

   Any linear transformation of the form \( f(x, y) = (ax + by, 2ax + by) \) has this property, where \( a, b \) are arbitrary real numbers. Thus, there are infinitely many such linear transformations.

6. Let \( S \) be the square with vertices \((0, 0), (1, 0), (0, 1), (1, 1)\). The number of linear transformations of \( \mathbb{R}^2 \) which map \( S \) to itself is:
   
   - 1
   - 2
   - 3
   - infinitely many.

   Let \( f \) be a linear transformation which maps \( S \) to itself. Then, each vertex of \( S \) maps to a vertex of \( S \). It is easy to see that \( f(0, 0) = (0, 0) \), and \( f(1, 1) \) can only be \((1, 1)\). Thus, there is only freedom in where \((1, 0)\) and \((0, 1)\) map to. We can either have \( f(1, 0) = (1, 0) \), \( f(0, 1) = (0, 1) \) or \( f(1, 0) = (0, 1) \), \( f(0, 1) = (1, 0) \). These correspond to the linear transformations \( f(x, y) = (x, y) \) and \( f(x, y) = (y, x) \).

7. Let \( a > 0 \) and define the linear transformation \( f(x, y) = (ax - y, ax + y) \). If \( f \) dilates areas of regions of \( \mathbb{R}^2 \) by a factor of 6, then the value of \( a \) is: \( 3 \)

   The matrix representation of \( f \) is \( \begin{pmatrix} a & -1 \\ a & 1 \end{pmatrix} \). Its determinant is \(-2a\). The area dilation factor of \( f \) is the absolute value of this determinant, which is \( 2a \) (since \( a > 0 \)). Thus \( 2a = 6 \), or \( a = 3 \).