Assignment 3: Solutions

1. The number of permutations of 1, 2, ..., 6 with cycle type 3 + 3 is (please enter only the final numerical answer): 40

Three letters for the first 3-cycle can be chosen in \( \binom{6}{3} \) ways. Once we fix them, the three letters in the second 3-cycle get fixed. Now the first set of three letters gives rise to two 3-cycles, similarly the second set of three letters gives two 3-cycles. Therefore the number of permutations of 3+3 type looks like \( \binom{6}{3} \cdot 2 \cdot 2 \). But we have counted \((abc)(def)\) and \((def)(abc)\) differently. Hence the answer is \( \frac{\binom{6}{3} \cdot 2 \cdot 2}{2} = 40 \)

2. The number of permutations of 1, 2, ..., 6 with cycle type 3 + 2 + 1 is: 120

First choose 3 numbers for the 3-cycle; this can be done in \( \binom{6}{3} = 20 \) ways. Arranging these 3 numbers into a cycle can be done in 2! ways. Next, from the remaining 3 numbers, choose 2 of them; this can be done in \( \binom{3}{2} = 3 \) ways; arranging them in a cycle can only be done in 1! = 1 way. There is no choice left for the 1-cycle. Thus, total number equals \( 20 \times 2 \times 3 = 120 \) ways.

3. Let \( \sigma \) be the following permutation of 1, 2, ..., 5 (written in cycle notation): \( \sigma = (1 \ 5)(2 \ 4)(3) \). The number of inversions (crossings) of \( \sigma \) is: 10

4. Let \( \pi \) be the following permutation of 1, 2, ..., 5 (written in cycle notation): \( \pi = (1 \ 3 \ 5)(2 \ 4) \). The number of inversions of \( \pi \) is: 7

5. Let \( n \) be a natural number, and let \( S_n \) denote the set of all permutations of 1, 2, ..., \( n \). What is the maximum possible number of crossings a permutation in \( S_n \) could have?

\[
\cdot \frac{n(n+1)}{2}.
\]

\[
\cdot \frac{n(n-1)}{2}.
\]

\[
\cdot n^2.
\]

\[
\cdot n.
\]

\( n, (n-1), ..., 2, 1 \) (the numbers written in descending order) is the unique permutation in which every pair \( i < j \) is an inversion.

6. Referring back to the previous question, how many different permutations in \( S_n \) have this maximum number of crossings?

\[
\cdot 1.
\]

\[
\cdot 2.
\]

\[
\cdot n!.
\]

\[
\cdot n.
\]
Refer the previous problem.

7. Let $\sigma, \pi$ be arbitrary permutations in $S_n$ (where $S_n$ is as above). Then $\sigma \circ \pi$ and $\pi \circ \sigma$ have the same number of crossings.

- True.
- False.

Find some easy examples in $S_3$. 