Solution of Assignment - 6

(1) Ans - (A)

Let $a_n = \dfrac{n^2}{3^n}$, $n \in \mathbb{N}$.

Now $\underset{n\to\infty}{\text{Lt}} \dfrac{a_{n+1}}{a_n} = \underset{n\to\infty}{\text{Lt}} \dfrac{(n+1)^2}{3^{n+1}} \cdot \dfrac{3^n}{n^2}$

$$= \underset{n\to\infty}{\text{Lt}} \frac{1}{3}\left(\frac{n+1}{n}\right)^2$$

$$= \frac{1}{3} < 1.$$

By Ratio test it is converegent.

(2) Ans - (B)

Let $a_n = \dfrac{x^n}{n}$, $n \in \mathbb{N}$ and $x > 1$.

Now $\underset{n\to\infty}{\text{Lt}} \dfrac{a_{n+1}}{a_n} = \underset{n\to\infty}{\text{Lt}} \dfrac{x^{n+1}}{n+1} \cdot \dfrac{n}{x^n}$

$$= \underset{n\to\infty}{\text{Lt}} \left(\frac{n}{n+1}\right) x$$

$$= x > 1.$$

Hence by Ratio ~~Root~~ test it is diveregent.

(3) Ans — (A)

Let $a_n = \left(n^{\frac{1}{n}} - 1\right)^n$.

Now $\underset{n\to\infty}{Lt}\, a_n^{\frac{1}{n}} = \underset{n\to\infty}{Lt}\left[\left(n^{\frac{1}{n}}-1\right)^n\right]^{\frac{1}{n}}$

$= \underset{n\to\infty}{Lt}\left(n^{\frac{1}{n}}-1\right)$

$= 1 - 1 = 0 < 1 \quad \left[\because \underset{n\to\infty}{Lt}\, n^{\frac{1}{n}} = 1\right]$

Hence by ~~Ratio~~ Root test it is convergent.

(4) Ans — (B)

A series $\sum_{n=1}^{\infty} a_n$ is said to be conditionally convergent if $\sum_{n=1}^{\infty} a_n$ ~~diverges~~ converges but $\sum_{n=1}^{\infty} |a_n| = \infty$. i.e, convergent but not absolutely convergent.

(5) Ans — (B)

Let $a_n = \sin\frac{1}{n} + \frac{1}{n^2}$, and $b_n = \frac{1}{n}$, $n \in \mathbb{N}$.

Now $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} n\left(\sin\frac{1}{n} + \frac{1}{n^2}\right)$

$= \lim_{n\to\infty} \left(n \sin\frac{1}{n} + \frac{1}{n}\right)$

$= 1 + 0 = 1$

Hence by comparison test (limit form) Σa_n and Σb_n converges or diverge simultaneously.

Since Σb_n diverges so Σa_n diverges.

(6) Ans :- (A)

Let $a_n = \frac{1}{(2n-1)(2n+1)}$ and $b_n = \frac{1}{n^2}$, $n \in \mathbb{N}$.

Now $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{(2n-1)(2n+1)}$

$= \lim_{n\to\infty} \frac{1}{\left(2-\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}$

$= \frac{1}{4}$.

Since Σb_n converges so Σa_n converges also.

(7) Ans - (A)

Let $a_n = 3^{-n + \frac{(-1)^n}{n}}$

Now $\lim_{n\to\infty} a_n^{\frac{1}{n}} = \lim_{n\to\infty} \left[3^{-n + \frac{(-1)^n}{n}} \right]^{\frac{1}{n}}$

$= \lim_{n\to\infty} 3^{-1 + \frac{(-1)^n}{n^2}}$

$= 3^{-1} < 1.$

Hence by Root test it is converegent.

(8) Ans : (B)

Let $a_n = \dfrac{1}{\sqrt{n} + \sqrt{n+1} + \sqrt{n+2} + \sqrt{n+3}}$ and $b_n = \frac{1}{\sqrt{n}}$, $n \in \mathbb{N}$.

Now $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \lim_{n\to\infty} \dfrac{\sqrt{n}}{\sqrt{n} + \sqrt{n+1} + \sqrt{n+2} + \sqrt{n+3}}$

$= \lim_{n\to\infty} \dfrac{1}{\sqrt{1} + \sqrt{1+\frac{1}{n}} + \sqrt{1+\frac{2}{n}} + \sqrt{1+\frac{3}{n}}}$

$= \frac{1}{4}.$

Since Σb_n diverges, so Σa_n diverges.

(9) Ans: (B)

Let $a_n = \dfrac{\left(1+\frac{1}{n}\right)^{n^2}}{x^n}$, $n \in \mathbb{N}$.

Now $\lim_{n\to\infty} a_n^{\frac{1}{n}} = \lim_{n\to\infty} \left[\dfrac{\left(1+\frac{1}{n}\right)^{n^2}}{x^n} \right]^{\frac{1}{n}}$

$= \lim_{n\to\infty} \dfrac{\left(1+\frac{1}{n}\right)^{n}}{x}$

$= \dfrac{e}{x}.$

Now if $x < e \Rightarrow \frac{e}{x} > 1 \Rightarrow \sum a_n$ diverceges

if $x > e \Rightarrow \frac{e}{x} < 1 \Rightarrow \sum a_n$ converceges

if $x < \frac{e}{2} \Rightarrow \frac{e}{x} > 2 \Rightarrow \sum a_n$ divereges

if $x = \frac{e}{2} \Rightarrow \frac{e}{x} = 2 \Rightarrow \sum a_n$ divereges.

(10) Ans - (B)

Since $\sum a_n$ is convergent $\Rightarrow \lim_{n\to\infty} a_n = 0$.

Hence $\lim_{n\to\infty} \dfrac{a_n + 1}{n^{\frac{1}{n}}} = \dfrac{1}{1} = 1$