(a) \( f(x) = \frac{1}{x} \) on \([0, 1]\)

Let \( x_n = \frac{1}{n} \) and \( y_n = \frac{1}{n+1} \)

Then \( \lim |x_n - y_n| = 0 \)

But \( |f(x_n) - f(y_n)| = |n - (n+1)| = 1 \)

So, not uniformly continuous.

(b) \( f(x) = \sin x^2 \) on \([0, \infty]\)

Let \( x_n = \sqrt{n\pi} \) and \( y_n = \sqrt{n\pi + \frac{\pi}{2}} \)

Then, \( \lim |x_n - y_n| = 0 \)

But \( |f(x_n) - f(y_n)| = |\sin n\pi - \sin (\frac{n\pi + \pi}{2})| = |\cos n\pi| = 1 \)

So, not uniformly continuous.

(c) \( |\sin x - \sin y| = |2\sin \left(\frac{x-y}{2}\right) \cos \left(\frac{x+y}{2}\right)| \)

\[ \leq |2\sin \left(\frac{x-y}{2}\right)| \leq |\frac{x(x-y)}{2}| \]

i.e. \( |\sin x - \sin y| \leq |x-y| \)

\( \Rightarrow \) \( f(x) = \sin x \) is Lipschitz function.

\( \Rightarrow \) \( f(x) = \sin x \) is uniformly continuous.
(2) (d) \( f(x) = x^2 \) on \([0, 00]\)

let \( x_n = \sqrt{n+1} \) and \( y_n = \sqrt{n} \)

then, \( \lim n |x_n - y_n| = 0 \)

But \( |f(x_n) - f(y_n)| = 1 \)

so, not uniformly continuous.

(3) (a) as from theory.

(4) (c) 

(5) (d) from Intermediate Value theorem.

(6) (d) Explained in lecture -52.
(7) (b) 
\[ f(x) = \begin{cases} 
0 & \text{when } x > 0 \\
2 & \text{when } x \leq 0 
\end{cases} \]

so, \( f \) has a discontinuity of first kind at \( x = 0 \).

(8) (a) 
\[ f(x) = \begin{cases} 
1 & \text{, when } x \text{ is irrational} \\
-1 & \text{, when } x \text{ is rational} 
\end{cases} \]

First, let \( a \) be any rational number so that \( f(a) = -1 \).

Since, in any interval there lie an infinite number of rational and irrational numbers, therefore, for each positive integer \( n \), we can choose an irrational number \( a_n \) so that \( |a_n - a| < \frac{1}{n} \).

Thus the sequence \( \{a_n\} \) converges to \( a \).

But \( f(a_n) = 1 \) for all \( n \), and \( f(a) = -1 \), so that
\[ \lim_{n \to \infty} f(a_n) \neq f(a). \]

Thus the function is discontinuous at any rational number \( a \).

Next, let \( b \) be any irrational number.
For each positive integer \( n \) we can choose a rational number \( b_n \) such that \( |b_n - b| < \frac{1}{n} \). Thus the sequence converges to \( b \).

But \( f(b_n) = -1 \) for all \( n \) and \( f(b) = 1 \).

\[ \lim_{n \to \infty} f(b_n) = f(b) \]

Hence, the function is discontinuous at all irrational points.

\(9\) \(a\)

\[ \lim_{x \to 0} \frac{\sin 2x}{x} = 2 \]

So, \( f \) has removable discontinuity at \( x = 0 \).

\(10\) \(c\)

\[ f(x) = \begin{cases} x, & \text{when } x \text{ is irrational} \\ -x, & \text{when } x \text{ is rational} \end{cases} \]

First, let \( a \neq 0 \) be any rational number, so that \( f(a) = -a \). Since in every interval there lie an infinite number of rational and irrational numbers, therefore, for each positive integer \( n \),
we can choose an irrational number \(\alpha\) such that

\[|a_n - \alpha| < \frac{1}{n}\]

Thus the sequence \(\{a_n\}\) converges to \(\alpha\)

\[\lim_{n \to \infty} f(a_n) = \lim_{n \to \infty} a_n = \alpha\]

Thus \(\lim_{n \to \infty} f(a_n) \neq f(\alpha), \ \alpha \neq 0\)

So, the function is discontinuous at any rational number, other than zero.

In a similar way, the function may be shown to be discontinuous at every irrational point.

Let \(\epsilon > 0\) be given. Then for \(\delta = \epsilon\), we have

\[|x| < \delta \Rightarrow |f(x) - f(\alpha)| = \left|\frac{x}{1-x}\right| < \epsilon, \quad \text{when} \ x \ \text{is rational}\]

and

\[|x| < \delta \Rightarrow |f(x) - f(\alpha)| = \epsilon - |x| < \epsilon, \quad \text{when} \ x \ \text{is irrational}\]

Thus

\[|x| < \delta \Rightarrow |f(x) - f(\alpha)| < \epsilon\]

Hence, the function is continuous at \(x = 0\).