Assignment - 9 solution

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(1) (c)

(a) $f(x) = \frac{1}{x}$ on $]0,1]$

let $x_n = \frac{1}{n}$ and $y_n = \frac{1}{n+1}$,

then $\lim |x_n - y_n| = 0$

But $|f(x_n) - f(y_n)| = |n - (n+1)| = 1$

So, not uniformly continuous.

(b) $f(x) = \sin x^2$ on $[0, \infty[$

let $x_n = \sqrt{n\pi}$ $\quad y_n = \sqrt{n\pi + \frac{\pi}{2}}$

then, $\lim |x_n - y_n| = 0$

But $|f(x_n) - f(y_n)| = |\sin n\pi - \sin(\frac{\pi}{2} + n\pi)|$

$= |\cos n\pi| = 1$

so, not uniformly continuous

(c) $|\sin x - \sin y| = \left|2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)\right|$

$\leq \left|2\sin\left(\frac{x-y}{2}\right)\right| \leq \left|\cancel{2}\frac{(x-y)}{\cancel{2}}\right|$

i.e. $|\sin x - \sin y| \leq |x - y|$

$\Rightarrow$ $f(x) = \sin x$ is Lipschitz function.

$\Rightarrow$ $f(x) \sin x$ is uniformly continuous.

(2) (d) $f(x) = x^2$ on $[0, \infty[$

let $x_n = \sqrt{n+1}$ and $y_n = \sqrt{n}$

then, $\lim |x_n - y_n| = 0$

But $|f(x_n) - f(y_n)| = 1$

So, not uniformly continuous.

(3) (a) st. from theory.

(4) (c)

(5) (d) from Intermediate value theorem.

(6) (d) Explained in lecture -52.

(7) (b)

$$f(x) = \begin{cases} 0 & \text{when } x > 0 \\ 2 & \text{when } x \leq 0 \end{cases}$$

So, f has a discontinuity of first kind at $x=0$.

(8) (a)

$$f(x) = \begin{cases} 1, & \text{when } x \text{ is irrational} \\ -1, & \text{when } x \text{ is rational} \end{cases}$$

First,

let a be any rational no so that $f(a) = -1$.

Since in any interval there lie an infinite number of rational and irrational numbers, therefore, for each positive integer n, we can choose an irrational number a_n s.t. $|(a_n - a)| < \frac{1}{n}$.

Thus the sequence $\{a_n\}$ converges to a.

But $f(a_n) = 1$ for all n, and $f(a) = -1$,

so that

$$\lim_{n \to \infty} f(a_n) \neq f(a).$$

Thus the function is discontinuous at any rational number a.

Next, let b be any irrational number.

For each positive integer n we can choose a rational number b_n such that $|b_n - b| < \frac{1}{n}$.

Thus the sequence converges to b.

But $f(b_n) = -1$ for all n and $f(b) = 1$.

$$\therefore \lim_{n \to \infty} f(b_n) \neq f(b)$$

Hence, the function is discontinuous at all irrational points.

(9) (a)

$$\lim_{x \to 0} \frac{\sin 2x}{x} = 2$$

So, f has removable discontinuity at $x=0$.

10) (c)

$$f(x) = \begin{cases} x, & \text{when } x \text{ is irrational} \\ -x, & \text{when } x \text{ is rational.} \end{cases}$$

First, let $a \neq 0$ be any rational number, so that $f(a) = -a$. Since in every interval there lies an infinite number of rational and irrational numbers, therefore, for each positive integer n, we can choose an irrational number a_n such that

$$|a_n - a| < \frac{1}{n}$$

Thus the sequence $\{a_n\}$ converges to a

$$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} a_n = a$$

Thus $\lim_{n\to\infty} f(a_n) \neq f(a)$, $a \neq 0$

So, the function is discontinuous at any rational number, other than zero.

In a similar way the function may be shown to be discontinuous at every irrational point.

let $\epsilon > 0$ be given. Then for $\delta = \epsilon$, we have

$$|x| < \delta \Rightarrow |f(x) - f(0)| = |-x| = |x| < \epsilon,$$

when x is rational

and $|x| < \delta \Rightarrow |f(x) - f(0)| = |x| < \epsilon$,

when x is irrational

Thus

$$|x| < \delta \Rightarrow |f(x) - f(0)| < \epsilon$$

Hence, the function is continuous at $x = 0$.