(1) $\lim\limits_{x \to \infty} x \sin\frac{1}{x}$

$$= \lim_{x \to \infty} \frac{\sin\frac{1}{x}}{\frac{1}{x}}$$

$$= \lim_{t \to 0} \frac{\sin t}{t} \quad \text{where } \frac{1}{x} = t$$

$$= 1.$$

(2) Given $f(x) = \sin\frac{1}{x}$, $x \in (0,1)$.

It can be easily seen that for any point $a \in (0,1)$ the limit, $\lim\limits_{x \to a} f(x)$ exists and equal to the value $f(a)$. So it is continuous.

(3)

(A) $\lim\limits_{x \to 0} \cos\frac{1}{x}$ does not exists. So it is not continuous at origin.

(B) $\lim\limits_{x \to 0} \left(x + \sin\frac{1}{x}\right)$ does not exists. So it is also not continuous at origin.

(c) $\lim_{x \to 0} (\sin x)(\sin \frac{1}{x})$

Since $\sin\frac{1}{x}$ is bounded in a deleted nbd of 0 so,

$$\lim_{x \to 0} \sin x \sin\frac{1}{x} = 0 = f(0).$$

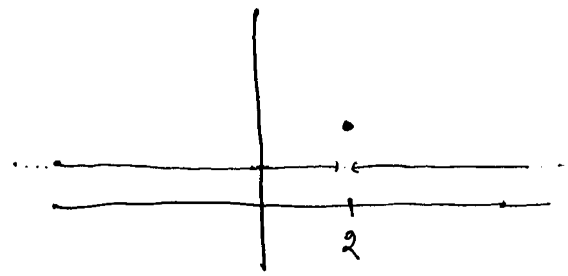
Hence it is continuous at origin.

(D) $\lim_{x \to 0} x \sin\frac{1}{x} = 0 \neq f(0)$.

Hence it is not continuous at origin.

(4) We know that if f and g are two functions and if they are continuous at point, then the product $f \cdot g$ is also continuous at that point.

(5) Given $f(x) = \begin{cases} 2, & x = 2 \\ 1, & x \neq 2. \end{cases}$



(A) $\lim_{x \to 2} f(x) = 1 \neq 2$.

(B) $\lim_{x \to 2} f(x) = 1 \neq f(2)$.

Hence it is not continuous.

(C) $\lim_{x \to 2} f(x)$ exists.

(D) $f(x)$ is discontinuous, it is discontinuous at $x = 2$.

(6) It is well known that if f and g are two functions defined on an interval and f, g are continuous at a point p in the interval, then the function $\frac{f}{g}$ is continuous at p if $g(p) \neq 0$.

(7) Given $f(x) = \begin{cases} \cos x, & x \geq 0 \\ -\cos x, & x < 0 \end{cases}$

It is clear that $f(x)$ is continuous when $x > 0$ and $x < 0$. We have to check at $x = 0$.

Now $\lim_{x \to 0^+} f(x) = \lim_{x \to 0} \cos x = 1.$

and $\lim_{x \to 0^-} f(x) = \lim_{x \to 0} (-\cos x) = -1.$

Hence the limit of $f(x)$ at $x \to 0$ does not exists. So it is not continuous at $x = 0$.

(8) Given $f(x) = \begin{cases} 2x, & 0 \le x < 1 \\ 3, & x = 1 \\ 4x, & 1 < x \le 2. \end{cases}$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1} 4x = 4,$

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1} 2x = 2.$

Hence $\lim_{x \to 1} f(x)$ does not exist. So $f(x)$ is not continuous at $x = 1$.

Also $f(x) = 2x$, for $x \in (0,1)$. So $f(x)$ is continuous on $(0,1)$.

(9) $\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{e^x - x - 1}{x^2}$ $\left(\frac{0}{0} \text{ form}\right)$.

$$= \lim_{x \to 0} \frac{e^x - 1}{2x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{e^x}{2} = \frac{1}{2}$$

$\therefore \lim_{x \to 0} f(x) \neq f(0)$.

So $f(x)$ is discontinuous at $x = 0$.

(10) $\lim_{n \to \infty} \sin\left(\pi \sqrt{\frac{1}{n^2} + 1}\right)$

$$= \sin\left[\lim_{n \to \infty} \pi \sqrt{\frac{1}{n^2} + 1}\right] \quad \left[\because \sin x \text{ is a continuous function}\right]$$

$$= \sin \pi = 0$$