Week 5: Assignment Solution

(1) (a) Let $u_n = \frac{n}{n+1}$, then

$$|u_n - u_m| = \left|\frac{n}{n+1} - \frac{m}{m+1}\right|$$

$$= \left|\frac{mn + n - mn - m}{(n+1)(m+1)}\right|$$

$$= \left|\frac{(n-m)}{(n+1)(m+1)}\right|$$

if $n \geq m$, then

$$|u_n - u_m| = \frac{(n-m)}{(n+1)(m+1)} < \frac{1}{m+1} < \frac{1}{m}$$

Let $\epsilon > 0$. Then $\frac{1}{m} < \epsilon$ holds for $m > \frac{1}{\epsilon}$.

Let $H = \left[\frac{1}{\epsilon}\right] + 1$. Then

$$|u_n - u_m| < \epsilon \quad \forall \quad n, m \geq H.$$

This proves that the sequence $\{u_n\}$ is a Cauchy sequence.

2 (d)

Let $\{x_n\}$ be a Cauchy sequence. Then by definition, to each $\epsilon > 0$, $\exists$ ~~a $\delta > 0$~~ ~~an~~ $n_0 \in \mathbb{N}$ s.t.

$$|x_n - x_m| < \epsilon \quad \forall\ n, m \geq n_0$$

Consequently,

$$|x_n - x_{n_0}| < \epsilon \quad \forall\ n \geq n_0$$

Now,

$$|x_n| = |x_n - x_{n_0} + x_{n_0}| \leq |x_n - x_{n_0}| + |x_{n_0}|$$
$$< \epsilon + |x_{n_0}| = \overline{M} \text{ (say)}$$
for all $n \geq n_0$.

Consider $M = \max^m \{|x_1|, |x_2|, \cdots |x_{n_0 - 1}|, \overline{M}\}$

Then $|x_n| \leq M \quad \forall\ n \in \mathbb{N}$.

So, $\{x_n\}$ is a bounded sequence.

(3) (b)

~~$\{S_n\}$~~ Here, $s_n = \begin{cases} 2, & \text{when } n \text{ is even} \\ \text{lowest prime factor } (\neq 1) \text{ of } n, & \text{when } n \text{ is odd.} \end{cases}$

Clearly the sequence is

$\{1, 2, 3, 2, 5, 2, 7, 2, 3, 2, 11, 2, 13, \ldots\}$

It is bounded below but unbounded above. So, from divergence criteria the given sequence (s_n) is divergent.

4 (a)

Let $x_n = \dfrac{3+2\sqrt{n}}{\sqrt{n}}$

then, $|x_n - x_m| = \left| \dfrac{3+2\sqrt{n}}{\sqrt{n}} - \dfrac{3+2\sqrt{m}}{\sqrt{m}} \right|$

$$= \left| \frac{3\sqrt{m} + \cancel{2\sqrt{mn}} - 3\sqrt{n} - \cancel{2\sqrt{mn}}}{\sqrt{nm}} \right|$$

$$= \left| \frac{3}{\sqrt{n}} - \frac{3}{\sqrt{m}} \right| \leq \frac{3}{\sqrt{n}} + \frac{3}{\sqrt{m}} \quad \text{—(1)}$$

Now, for any $\epsilon > 0$, we can find a natural number n_0 s.t.

$$\frac{1}{\sqrt{n}} < \frac{\epsilon}{6} \;\; \forall \;\; n \geq n_0$$

then from (1)

$$|x_n - x_m| \leq \frac{3}{\sqrt{n}} + \frac{3}{\sqrt{m}} < 3\left(\frac{\epsilon}{6} + \frac{\epsilon}{6}\right) = \epsilon$$

for all $n, m \geq n_0$

So, $\left\{ \dfrac{3+2\sqrt{n}}{\sqrt{n}} \right\}$ is a Cauchy sequence.

5 (c)

$$x_{n+1} = \frac{b_{n+1}}{b_{n+2}} = \frac{b_{n+1}}{b_{n+1}+b_n} = \frac{1}{1+\frac{b_n}{b_{n+1}}} = \frac{1}{1+x_n}$$

. An induction argument establishes

$\frac{1}{2} \leq x_n \leq 1$ for all n.

So adding 1, we have

$$\frac{3}{2} \leq x_n + 1 \leq 2$$

$$\Rightarrow \frac{1}{2} \leq \frac{1}{x_n+1} \leq \frac{2}{3} \text{ for all } n.$$

Now, $|x_{n+1} - x_n| = \left|\frac{1}{1+x_n} - \frac{1}{1+x_{n-1}}\right|$

$$= \frac{|x_n - x_{n-1}|}{(1+x_n)(1+x_{n-1})} \leq \frac{2}{3}\cdot\frac{2}{3}\,|x_n - x_{n-1}|$$

$$= \frac{4}{9}\,|x_n - x_{n-1}|$$

Hence, the sequence $\{x_n\}$ is contractive.

6 (b)

let $\epsilon = \frac{\ln 2}{2}$

take $m = 2n$

Then, $|x_m - x_n| = |x_{2n} - x_n|$

$= |\ln 2n - \ln n|$

$= \left|\ln \frac{2n}{n}\right|$

$= |\ln 2| > \epsilon$

$\therefore$ $(\ln n)$ is not a cauchy sequence.

7 (d)

$\cos\frac{1}{n} \not\to 0$, $\cos\frac{1}{\sqrt{n}} \not\to 0$, $\cos\frac{1}{n^2} \not\to 0$

So, $\sum_{n=1}^{\infty} \cos\frac{1}{n}$, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ and $\sum_{n=1}^{\infty} \cos\frac{1}{n^2}$ are

not convergent

8 (b) straight from lectures

9 (b) st. from Pringsheim's theorem.

10) (d) $\ln(n)$; e^n, and $\frac{1}{\sqrt{n}}$ are not always integers. But $n!$ is always an integer.

So $\left(\sin\frac{1}{n!}\right)$ is a subsequence of

$\left(\sin\frac{1}{n}\right)$