~~Set~~ Solution of Week 4 Assignment

① If $\{S_n\}$ is a sequence s.t $S_{n+2} = \frac{1}{2}(S_{n+1} + S_n)$ for $n \geq 1$ and $0 < S_1 < S_2$, then the sequence $\{S_n\}$ converges to $\frac{S_1 + 2S_2}{3}$.

Solⁿ

$$S_2 - S_1 > 0, \quad S_3 - S_2 = \frac{1}{2}(S_2 + S_1) - S_2 = -\frac{1}{2}(S_2 - S_1)$$

$$S_4 - S_3 = \frac{1}{2}(S_3 + S_2) - S_3 = \frac{1}{2}(S_2 - S_3) = \left(-\frac{1}{2}\right)^2 (S_2 - S_1)$$

$$\cdots\cdots$$

$$S_n - S_{n-1} = \left(-\frac{1}{2}\right)^{n-2} (S_2 - S_1)$$

$$\therefore \quad S_n - S_1 = (S_2 - S_1)\left[1 + \left(-\frac{1}{2}\right) + \left(-\frac{1}{2}\right)^2 + \cdots + \left(-\frac{1}{2}\right)^{n-2}\right]$$

$$= \frac{2(S_2 - S_1)}{3}\left[1 - \left(-\frac{1}{2}\right)^{n-1}\right]$$

Now $\lim\limits_{n\to\infty} (S_n - S_1) = \frac{2}{3}(S_2 - S_1)$,

Since $\lim\limits_{n\to\infty} \left(-\frac{1}{2}\right)^{n-1} = 0$ therefore $\lim\limits_{n\to\infty} S_n = S_1 + \frac{2}{3}(S_2 - S_1)$

$$= \frac{S_1 + 2S_2}{3}.$$

② $$\lim_{n\to\infty} \frac{\{(n+1)(n+2)\cdots(2n)\}^{\frac{1}{n}}}{n} = \frac{4}{e}$$

Solⁿ Let $U_n = \dfrac{(n+1)(n+2)\cdots 2n}{n^n}$.

Then $U_n > 0$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} \dfrac{U_{n+1}}{U_n} = \dfrac{2(2n+1)}{(n+1)} \cdot \dfrac{1}{\left(1+\frac{1}{n}\right)^n}$

$$= \frac{4}{e} > 0$$

∴ Hence by Cauchy 2ⁿᵈ theorem on limit

$$\lim_{n\to\infty} U_n^{\frac{1}{n}} = \lim_{n\to\infty} \frac{\{(n+1)(n+2)\cdots(2n)\}^{\frac{1}{n}}}{n} = \frac{4}{e}.$$

③ $\lim_{n\to\infty} \dfrac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{n} = 0$

Soln Let $U_n = \frac{1}{n}$. Then $\lim_{n\to\infty} U_n = 0$.

By Cauchy first theorem on limit

$$\lim_{n\to\infty} \frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{n} = 0.$$

④ $\lim_{n\to\infty} \dfrac{2^n}{n} = \infty$

Soln $a_n = \frac{2^n}{n}$. Clearly $a_n \neq 0 \;\forall n \in \mathbb{N}$ and positive

Now $\dfrac{a_{n+1}}{a_n} = \dfrac{2^{n+1}}{(n+1)} \cdot \dfrac{n}{2^n} = \dfrac{2.\, n}{(n+1)} = \dfrac{2}{(1+\frac{1}{n})}$

$\therefore \lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} = 2 > 1$

Hence $\lim_{n\to\infty} \dfrac{2^n}{n} = \infty$ [By Ratio Test]

⑤ $\lim_{n\to\infty} \dfrac{(3n+1)(n-2)}{n(n+3)} = 3$

Soln

$$\lim_{n\to\infty} \frac{(3n+1)(n-2)}{n(n+3)} = \lim_{n\to\infty} \frac{n^2(3+\frac{1}{n})(1-\frac{2}{n})}{n^2(1+\frac{3}{n})}$$

$$= \lim_{n\to\infty} \frac{(3+\frac{1}{n})(1-\frac{2}{n})}{(1+\frac{3}{n})} = 3.$$

⑥ If $\lim_{n\to\infty} \frac{x^n}{n} = \infty$ then $x > 1$

Sol^n

If $x = 1$ then $\lim_{n\to\infty} \frac{x^n}{n} = \lim_{n\to\infty} \frac{1}{n} = 0$

Now $S_n = \frac{x^n}{n}$ then $\frac{S_{n+1}}{S_n} = \frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n}$

$= x \cdot \frac{n}{n+1}$

$= x \cdot \frac{1}{1+\frac{1}{n}}$

$\therefore \lim_{n\to\infty} \frac{S_{n+1}}{S_n} = \lim_{n\to\infty} \frac{x}{\left(1+\frac{1}{n}\right)}$

$= x.$

So by Ratio test if $x < 1$ then $\lim_{n\to\infty} \frac{x^n}{n}$ convergent

and if $x > 1$ then $\lim_{n\to\infty} \frac{x^n}{n} = \infty$

So if $\lim_{n\to\infty} \frac{x^n}{n} = \infty$ then $x > 1$.

⑦ $\lim_{n\to\infty} \frac{(-1)^n n^3 + 1}{2n^3 + 3}$ doesnot exist

Sol^n

$S_n = \frac{(-1)^n n^3 + 1}{2n^3 + 3}$

$\therefore S_n = \frac{n^3 + 1}{2n^3 + 3}$ ~~for~~ ~~n is even~~ when n is even

and $S_n = \frac{-n^3 + 1}{2n^3 + 3}$ ~~for~~ when n is odd.

Now $\{S_{2n}\} = \left\{ \frac{(2n)^3 + 1}{2 \cdot (2n)^3 + 3} \right.$ for $n = 1, 2, 3 \ldots \}$

and $\{S_{2n-1}\} = \left\{ -\frac{(2n-1)^3 + 1}{2(2n-1)^3 + 3} \right.$ for $n = 1, 2, 3 \ldots \}$

are two subsequence of $\{S_n\}$

S.t $\lim_{n\to\infty} S_{2n} = \frac{1}{2}$ and $\lim_{n\to\infty} S_{2n-1} = -\frac{1}{2}$

$\therefore \lim_{n\to\infty} \frac{(-1)^n n^3 + 1}{2n^3 + 3}$ doesnot exist

(8) $\lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) = 0$

Sol^n

$$\lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) = \lim_{n\to\infty} \frac{(n+1) - n}{\sqrt{n+1} + \sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

$$= 0.$$

(9) If $\{a_n\}$ converges and $\{b_n\}$ diverges to $+\infty$ then $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$

Sol^n

Since sequence $\{b_n\}$ diverges to $+\infty$ so $\lim_{n\to\infty} \frac{1}{b_n} = 0$.

Now $\lim_{n\to\infty} a_n$ converges and $\lim_{n\to\infty} \frac{1}{b_n} = 0$

$\therefore \lim_{n\to\infty} a_n \cdot \frac{1}{b_n} = 0.$

(10). If $a_n = \frac{(3n)!}{(n!)^3}$, then $\lim_{n\to\infty} a_n^{\frac{1}{n}} = 27$.

Sol^n

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left[\frac{(3n+3)!}{[(n+1)!]^3} \times \frac{(n!)^3}{(3n)!} \right]$$

$$= \lim_{n\to\infty} \frac{(3n+3)(3n+2)(3n+1)}{(n+1)^3}$$

$$= \lim_{n\to\infty} \frac{(3+\frac{3}{n})(3+\frac{2}{n})(3+\frac{1}{n})}{(1+\frac{1}{n})^3}$$

$$= 27.$$

$\therefore$ By cauchy 2^nd theorem on limit

$$\lim_{n\to\infty} a_n^{\frac{1}{n}} = 27.$$