Answer sheet of Assignment – 3

① In the set of real numbers, the set $(a,b) \subset \mathbb{R}$ is not compact.

Sol^n

Since (a,b) is open set in $\mathbb{R}$, so by Heine Borel theorem it is not a compact set.

② In the set of real numbers, the set $[0,\infty) \subset \mathbb{R}$ is closed but not bounded.

Sol^n

Since all the limit points of the set $[0,\infty)$ lies in $[0,\infty)$ so $[0,\infty)$ is closed in $\mathbb{R}$. Also it is clear that $[0,\infty)$ is not bounded in $\mathbb{R}$.

③ Suppose A is a bounded set in $\mathbb{R}$, then the closure of A is a compact set in $\mathbb{R}$.

Sol^n

Since A is bounded so $\bar{A}$ is also bounded. Again $\bar{A}$ is a closed set. Hence by Heine-Borel theorem $\bar{A}$ is compact.

④ The sequence $\{s_n\}$, where $s_n = 1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \cdots + \frac{1}{\lfloor n}$ is ~~[illegible]~~ bounded above and increasing.

Sol^n

we have $s_{n+1} - s_n = \frac{1}{\lfloor n+1} > 0 \quad \forall n \in \mathbb{N}$

Hence $\{s_n\}$ is monotonically increasing.

Now, $$s_n = 1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \cdots + \frac{1}{\lfloor n}$$
$$\leq 1 + \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}} \quad [\because \lfloor n \geq 2^{n-1} \; \forall n \in \mathbb{N}]$$
$$= 1 + \frac{1-(\frac{1}{2})^n}{1-\frac{1}{2}}$$
$$= 1 + 2\left(1-\left(\tfrac{1}{2}\right)^n\right) = 3 - \frac{1}{2^{n-1}} < 3.$$

⑤ The derive set of the ~~sequence~~ sequence $\{S_n\}$, where $S_n = \sin\frac{n\pi}{2} + \frac{(-1)^n}{n}$, $n \in \mathbb{N}$ is $\{1, 0, -1\}$

Solⁿ

$$a_n = \begin{cases} 1 + \frac{(-1)^n}{n} & ; \quad n = 4m-3, m \in \mathbb{N} \\ 0 + \frac{(-1)^n}{n} & ; \quad n = 2m, m \in \mathbb{N} \\ -1 + \frac{(-1)^n}{n} & ; \quad n = 4m-1, m \in \mathbb{N} \end{cases}$$

which shows that 1, 0, −1 are the only limit points of the set $\{S_n\}$. So the derive set of the set $\{S_n\}$ is $\{1, 0, -1\}$.

⑥ The sequence $\{S_n\}$, where $S_n = 1 + (-1)^n$, $n \in \mathbb{N}$ is periodic with period 2.

Solⁿ

Since $S_n = 1 + (-1)^n$, $n \in \mathbb{N}$

So, $S_n = 0$, when n is odd

$= 2$, when n is even.

So, $\{S_n\}$ is periodic with period 2.

⑦ $\lim\limits_{n\to\infty} \frac{3 + 2\sqrt{n}}{\sqrt{n}} = 2$

Solⁿ

Let $\varepsilon > 0$

$\therefore \left| \frac{3+2\sqrt{n}}{\sqrt{n}} - 2 \right| < \varepsilon$ if $\left| \frac{3}{\sqrt{n}} \right| < \varepsilon$ or $n > \frac{9}{\varepsilon^2}$.

Let m be a positive integer greater than $\frac{9}{\varepsilon^2}$.

Thus every $\varepsilon > 0$, $\exists$ a positive m, such that

$\left| \frac{3+2\sqrt{n}}{\sqrt{n}} - 2 \right| < \varepsilon$, whenever $n \geq m$

$\therefore \lim\limits_{n\to\infty} \frac{3+2\sqrt{n}}{\sqrt{n}} = 2$