Week 12 Assignment Solution

(1) (b) $\int_0^2 f(x)\,dx = \int_0^1 x^2\,dx + \int_1^2 x^3\,dx$

$$= \frac{1}{3} + \frac{16}{4} - \frac{1}{4}$$

$$= \frac{1}{3} + \frac{15}{4}$$

$$= \frac{\cancel{19}\,4 + 45}{12}$$

$$= \frac{49}{12} = 4\frac{1}{12}$$

$$\int_0^2 f(x)\,dx = \int_0^1 x^3\,dx + \int_1^2 x^2\,dx$$

$$= \frac{1}{4} + \frac{8}{3} - \frac{1}{3}$$

$$= \frac{1}{4} + \frac{7}{3}$$

$$= \frac{3 + 28}{12} = \frac{31}{12} = 2\frac{7}{12}$$

② (c) $\int_0^1 f(x)\,dx = \int_0^c f(x)\,dx + \int_c^1 f(x)\,dx$

$= \int_0^c c\,dx + \int_c^1 2c\,dx$

$= c\int_0^c dx + 2c\int_c^1 dx$

$= c^2 + 2c(1-c)$

$= c^2 + 2c - 2c^2$

$= \cancel{c^2}\ 2c - c^2$

so, $2c - c^2 = \frac{7}{16}$

$\Rightarrow 32c - 16c^2 - 7 = 0$

$\Rightarrow 16c^2 - 32c + 7 = 0$

$\Rightarrow 16c^2 - 4c - 28c + 7 = 0$

$\Rightarrow 4c(4c-1) - 7(4c-1) = 0$

$\Rightarrow (4c-7)(4c-1) = 0$

$\Rightarrow c = \frac{7}{4}, \frac{1}{4}$

But $c \in [0,1]$

$\therefore c = \frac{1}{4}$

Corollary → If a function that is monotonic on the interval $I = [a, b]$ and α is continuous and monotonically increasing on I, then $f \in R(\alpha)$.

Q(3) Let $f(x) = \sin x$ on $[0, \frac{\pi}{2}]$ and $\alpha(x) = x^2$ on $[0, \frac{\pi}{2}]$. Then

(1) $f \in R(\alpha)$

(2) $f \notin R(\alpha)$

(3) If $\beta(x) = x^3$ on $[0, \frac{\pi}{2}]$, then $f \in R(\alpha)$ but $f \notin R(\beta)$

(4) none of the above.

Note → Q no. 4 and 5 directly follows from theorems.

4) → (c)

(5) → (b)

(6) (a) $\int_0^2 f\, d\alpha = \int_0^2 f(x)\, \alpha'(x)\, dx$

$= \int_0^2 x^2 \; 2x\, dx$

$= 2\int_0^2 x^3\, dx$

$= \frac{2}{4} \cancel{16}\, \cancel{2}4$

$= \cancel{4}\; 8$

(7) (c) $\int_0^2 f\, d\alpha = \int_0^2 f(x)\, \alpha'(x)\, dx$

$= \int_0^2 [x]\; 2x\, dx$

$= 2\left(\int_0^1 x[x]\, dx + \int_1^2 x[x]\, dx\right)$

$= 0 + 2\int_1^2 x\, dx$

$= 2\left(\frac{4}{2} - \frac{1}{2}\right)$

$= 3$

First we check the option (c)

(8)(c) Let $P = \{0, x_1, x_2, 1\}$ be a partition of $[0,1]$ such that

$$\frac{2}{\pi} \in [x_1, x_2].$$

Now, If $\frac{2}{\pi} \le x \le 1$, then $\sin\frac{1}{x} \ge \sin 1$

and hence

$$U(P,f) \ge \sin 1 \left(1 - \frac{2}{\pi}\right) > 0$$

It follows that

$$\underline{\int} f\,dx = 0 \text{ and } \overline{\int} f\,dx \ge \sin 1 \left(1 - \frac{2}{\pi}\right) > 0$$

Thus f is not Riemann integrable.

(a) ~~is~~ and (b) are R-integrable because they are continuous.

(9) (b) because (b) is unbounded.

(c) is continuous and (a) has only one point of discontinuity.

(10) (a)

$$f(x) = \begin{cases} 1, & \text{when } \frac{1}{a} < x \le 1 \\ \frac{1}{a} & " \quad \frac{1}{a^2} < x \le \frac{1}{a} \\ \frac{1}{a^2} & " \quad \frac{1}{a^3} < x \le \frac{1}{a^2} \\ \frac{1}{a^3} & " \quad \frac{1}{a^4} < x \le \frac{1}{a^3} \\ \vdots & \\ 0 & , \text{when } x = 0 \end{cases}$$

$$\int_0^1 f\,dx = \int_{1/a}^{1} f\,dx + \int_{1/a^2}^{1/a} f\,dx + \int_{1/a^3}^{1/a^2} f\,dx + \cdots$$

$$= \int_{1/a}^{1} dx + \int_{1/a^2}^{1/a} \frac{1}{a}\,dx + \int_{1/a^3}^{1/a^2} \frac{1}{a^2}\,dx + \cdots$$

$$= \left(1 - \frac{1}{a}\right) + \frac{1}{a}\left(\frac{1}{a} - \frac{1}{a^2}\right) + \frac{1}{a^2}\left(\frac{1}{a^2} - \frac{1}{a^3}\right) + \cdots$$

$$= \left(1 - \frac{1}{a}\right) + \frac{1}{a^2}\left(1 - \frac{1}{a}\right) + \frac{1}{a^4}\left(1 - \frac{1}{a}\right) + \cdots$$

$$= \left(1 - \frac{1}{a}\right)\left(1 + \frac{1}{a^2} + \frac{1}{a^4} + \cdots\right)$$

$$= \left(\frac{a-1}{a}\right)\left(\frac{1}{1 - \frac{1}{a^2}}\right) = \frac{(a-1)}{\cancel{a}} \cdot \frac{a^{\cancel{2}}}{a^2 - 1}$$

$$= \frac{a}{a+1}$$