Solution of Assignment-11.

① Let us consider the function

$$f(x) = \frac{a_0}{n+1} x^{n+1} + \frac{a_1}{n} x^n + \cdots + a_n x.$$

Then $f(0) = 0$ and let $f(1) = \frac{a_0}{n+1} + \frac{a_1}{n} + \cdots + a_n = 0$.

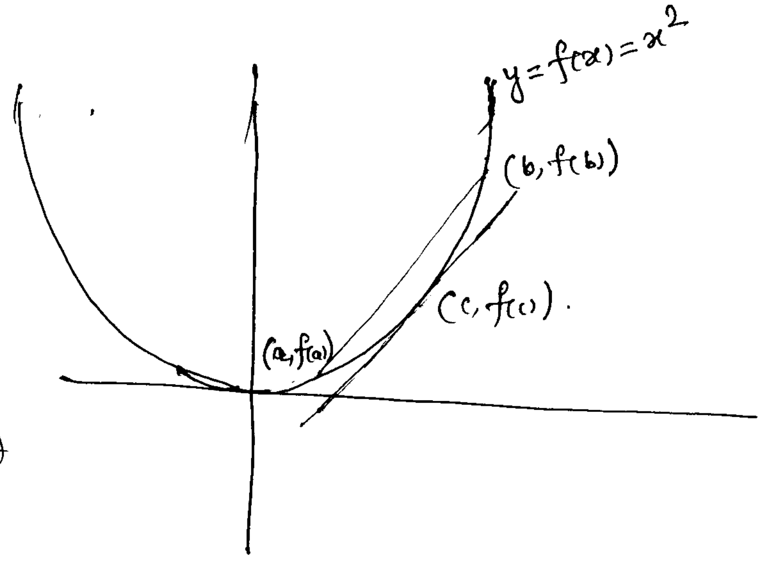
Then by Rolle's theorem we have, there exists a $c \in (0,1)$ such that

$$f'(c) = 0$$

$$\Rightarrow a_0 c^n + a_1 c^{n-1} + \cdots + a_n = 0.$$

$\Rightarrow a_0 x^n + a_1 x^{n-1} + \cdots + a_n = 0$ has at least one root between 0 and 1.

② By Lagrange's MVT we know that for $f(x) = x^2$, $x \in [a,b]$, there exists a point $c \in (a,b)$ such that



the chord joining $(a, f(a))$ and $(b, f(b))$ is parallel to the tangent at $(c, f(c))$

i.e, $\dfrac{f(b)-f(a)}{b-a} = f'(c)$.

$\Rightarrow \dfrac{b^2-a^2}{b-a} = 2c$.

$\Rightarrow c = \dfrac{a+b}{2}$.

$\therefore$ a, c, b are in A.P.

(3) We know that if $f(x)$ is a continuous function in $[a, b]$ and $f'(x) > 0$ in (a, b) then f is strictly increasing in $[a, b]$.

Here $f(x) = x^5 - x - 1$, $x \in [1, 2]$.

$\Rightarrow f'(x) = 5x^4 - 1 > 0 \;\forall\; x \in (1, 2)$.

$\Rightarrow f(x)$ is strictly increasing in $[1, 2]$.

(4) $\lim\limits_{x\to -4} \dfrac{\sin(\pi x)}{x^2-16}$ ($\frac{0}{0}$ form). Using L'Hospital rule

$= \lim\limits_{x\to -4} \dfrac{\pi\cos(\pi x)}{2x} = \dfrac{\pi \cos(4\pi)}{-8} = \dfrac{\pi}{-8}$

⑤ $\operatorname*{Lt}_{x \to 0} \dfrac{\sin(2x) + 7x^2 - 2x}{x^2 (x+1)^2}$ $\left(\frac{0}{0}\right)$, using L'Hospital rule

$$= \operatorname*{Lt}_{x \to 0} \frac{2\cos(2x) + 14x - 2}{2x(x+1)^2 + 2x^2(x+1)} \quad \left(\tfrac{0}{0} \text{ form}\right)$$

$$= \operatorname*{Lt}_{x \to 0} \frac{-4\sin(2x) + 14}{2(x+1)^2 + 4x(x+1) + 4x(x+1) + 2x^2}$$

$$= \frac{14}{2} = 7.$$

⑥ $\operatorname*{Lt}_{x \to \infty} (e^x + x)^{\frac{1}{x}}$

Let $y = (e^x + x)^{\frac{1}{x}}$

$\Rightarrow \ln y = \frac{1}{x} \ln(e^x + x).$

Now $\operatorname*{Lt}_{x \to \infty} \ln y = \ln\left(\operatorname*{Lt}_{x \to \infty} y\right).$

$\therefore \operatorname*{Lt}_{x \to \infty} \ln y = \operatorname*{Lt}_{x \to \infty} \dfrac{\ln(e^x + x)}{x}$ $\left(\frac{\infty}{\infty} \text{ form}\right).$

$$= \lim_{x\to\infty} \frac{e^x+1}{e^x+x} \quad \left(\frac{\infty}{\infty} \text{ form}\right).$$

$$= \lim_{x\to\infty} \frac{e^x}{e^x+1} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x\to\infty} \frac{e^x}{e^x} = 1.$$

$$\therefore \lim_{x\to\infty} y = e.$$

(7) $\boxed{K=1}$ Let $f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ x, & 1 < x \le 2. \end{cases}$

This is continuous, $f(0)=1$, $f(1)=1$, $f(2)=2$.
But $f(x)=\frac{1}{2}$ has no solution.

$\boxed{K=2}$ Let $f(x) = \begin{cases} 1+x, & 0 \le x \le 1 \\ 2, & 1 \le x \le 2. \end{cases}$

This is continuous, $f(0)=1$, $f(1)=2$, $f(2)=2$.
But $f(x)=\frac{1}{2}$ has no solution.

$\boxed{K=0}$ Then $f(0)=1$, $f(1)=0$, $f(2)=2$.

Now from intermediate value theorem, $f(x)$ attains all values between 0 and 1 in the interval $[0,1]$ and also f attains all values between 0 & 2 in the interval $[1,2]$. So $f(x)=\frac{1}{2}$ at least two solⁿ.

(8) Given $f(x)=\sin x$ has Taylor series expansion about $\frac{\pi}{2}$ of the form

$$a_0 + a_1\left(x-\frac{\pi}{2}\right) + a_2\left(x-\pi/2\right)^2 + \cdots$$

Then $$a_3 = \left.\frac{f'''(x)}{3!}\right|_{x=\pi/2} = \frac{-\cos(\pi/2)}{3!} = 0.$$

$$a_4 = \left.\frac{f^{iv}(x)}{4!}\right|_{x=\pi/2} = \frac{\sin(\pi/2)}{4!} = \frac{1}{24}$$

(9) Let $P: 0 = x_0 < x_1 < \cdots < x_n = 1$ be any partition of $[0,1]$. Also let for $r = 1, 2, \ldots, n$

$$M_r = \sup f(x), \quad x \in [x_{r-1}, x_r]$$

$$m_r = \inf f(x), \quad x \in [x_{r-1}, x_r].$$

Then $M_r = 1, \quad m_r = 0.$

$$\therefore L(P, f) = m_1(x_1 - x_0) + m_2(x_2 - x_1) + \cdots + m_n(x_n - x_{n-1})$$

$$= 0,$$

$$U(P, f) = M_1(x_1 - x_0) + M_2(x_2 - x_1) + \cdots + M_n(x_n - x_{n-1})$$

$$= 1.$$

(10) Answer $\Rightarrow$ $m(b-a) \leq M(b-a)$.